

THE APPLICATION OF  
MULTIVARIABLE OPTIMAL CONTROL  
TO  
NON-LINEAR CHEMICAL PROCESSES

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## CONTENTS

	PAGE
ABSTRACT	1
CHAPTER	
I INTRODUCTION	3
II MODERN CONTROL TECHNIQUES	17
III DYNAMIC PROGRAMMING	33
IV FEEDFORWARD CONTROL TECHNIQUES	49
V OPTIMAL FEEDBACK CONTROL OF LINEAR SYSTEMS	59
VI CONTROL OF NON-LINEAR STIRRED-TANK REACTORS	83
VII EXOTHERMIC NON-LINEAR REACTORS	103
VIII DISCUSSION AND CONCLUSIONS	127
REFERENCES	136
APPENDIX	
I CONTINUOUS SYSTEMS MODELLING PROGRAM	140
II SYSTEMS SUBROUTINES FOR MULTIVARIABLE OPTIMAL CONTROL	147

## ABSTRACT

Higher production rates, more extreme processing conditions, tighter product specifications and more highly integrated processing plant are envisaged as some of the reasons for encouraging the chemical industries to look more closely at the potential advantages of Modern Control Techniques. The advent of the process control computer has failed to bring about any significant shift away from the established conventional control techniques, perhaps because of the innate conservatism of the control engineers but more likely due to the apparent complexity of the mathematical techniques involved in so-called "Optimal" control, and a lack of confidence in the ability of optimal controllers to perform significantly better than the already highly-developed single-loop controllers.

The aims of this study are two-fold:

- (1) To demonstrate straightforward techniques for the solution of the optimisation problems which are the basis of optimal control theory, and
- (2) To demonstrate, by implementation of the control laws so obtained, some of the advantages which can accrue from multivariable optimal control.

Some of the standard multivariable control techniques are introduced and Dynamic Programming is selected for further development because of its versatility as an optimisation technique. The recurrence relationship is established for staged processes and the judicious use of some simplifying assumptions results in an iterative technique which converges rapidly to the solution of the steady-state control law. The Dynamic Programming approach is equally applicable to the optimisation of continuous processes and results in the Hamilton-Jacobi equation. Once again the use of simplifying assumptions leads to a straightforward method of solution and a steady-state multivariable control law.

The control laws are tested on a range of linear and non-linear systems and their performance compared with that of single- and multi-loop controllers under conditions of bounded controls, random disturbances and process and control variable "dead-time". The particular advantages of multivariable controllers are found to be

(1) More effective control, as measured by the process control criterion,

(2) An ability to stabilise the process under more severe disturbances,

(3) Greater process stability in the face of process or control variable "dead-time",

(4) The use of significantly less control effort in controlling the process, and

(5) The ability to control naturally unstable processes with tighter limits on the control variables.

Multivariable feedforward controllers are also developed and are shown to have significant advantages even on single-stage processes, although the quality of the process model is shown to be important. A non-linear feedforward compensator is seen to possess quite dramatic load rejection potential.

The multivariable optimal controllers are thus seen to be reasonably simple to implement, robust in operation and very effective. Possibly the greatest single advantage of multivariable over multi-loop control strategies is the elimination, at a stroke, of the configuration problem.

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# CHAPTER I

## INTRODUCTION

"Between the idea and the reality falls  
the shadow ..."

T.S. Eliot

The current status of Control Science is reviewed in relation to the Chemical Processing Industries and the split between the established control practices and the modern control theories is analysed.

	Page
Introduction	4
1.1 Single-Loop Control	7
1.2 Multivariable Control	8
1.3 Conventional Multi-Loop Technology	10
1.4 Automatic Measurement and Analytical Techniques	11
1.5 Process Control Computers	12
1.6 Chemical Process Plant	12
1.7 Modern Control Theory	13
1.8 Comparison of Multivariable and Multi-Loop Control Strategies	14

## CHAPTER 1

### INTRODUCTION

One of the first examples of the application of automatic control was the centrifugal governor designed by James Watt to control the speed of a steam engine. No longer did an operator have to concentrate slavishly on the machine's speed to control the steam pressure with a hand valve, and neither did the machine's performance depend so entirely upon the concentration, skill or whim of the human link in the feedback chain. Considerable progress has been made in all areas of automatic control over the intervening two hundred years if we consider the huge modern processing complexes controlled automatically under the almost casual supervision of a few men. The progress of control science has by no means been steady, however, and neither, as we shall see, has there been unanimity within the ranks of control engineers as to the best means of achieving its objectives. The mathematics and application of optimisation and automatic control received a boost during the latter years of the second world war, much of which did not see the light of day. New theories were taking shape, however, and once their interest had been kindled, people took their experience and skills into other areas. In more recent times, but in much the same way, the "spin offs" from the aerospace industry have resulted in a flood of new technologies.

The post-war years saw a steadily increasing interest in control theory and practice, particularly the systematic development of the Calculus of Variations as a technique for process optimisation. The greatest boost came in the late 1950's and early 1960's largely because of two developments. These were the very rapid advances being made in computing equipment and techniques, and the consolidation of variational methods - loosely known as Modern Control Theory. 1962 saw one of the landmarks in control theory and application with the first publication in English of the pioneering work of Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko, "The Mathematical Theory of Optimal processes". With Bellman's works, "Adaptive Control Processes" and "Applied Dynamic Programming", the future of Modern Control Theory was assured. A flood of books and papers was to follow, but worthy of mention in this context are Tou (1963, 1964), Lapidus and Luus

(1967), Noton (1965) and the most widely referenced book of all by Athens and Falb (1966).

Within the chemical industry developments were steadily changing the nature of chemical plant and, indeed, the processes themselves. Spurring these developments along was the rapid growth in the output of chemical industry and in the diversity of its products, with the consequent increase in new processing installations. Plants were increasing in size and throughput with each new generation bringing a huge increase in productivity. The economies of scale gave considerable impetus to the installation and improvement of process control equipment. Batch plant was giving way to continuous processes, especially where increased demand virtually excluded the possibility of economic batch processing. With the shift to continuous processes came the trend toward more integrated plant. No longer were individual plant items independent of each other, with large storage vessels acting as buffers to isolate them from upstream or downstream emergencies. Processes became more complex, with several "unit operations" sometimes incorporated into a single piece of equipment. A factor related to the change toward more integrated plant was the increase in processing "rate". Older plant was being pushed to its maximum capacity (and beyond) and new plant designed to increase throughput with the minimum physical plant size. Residence times became shorter and processing conditions more extreme, and the performance of control systems rose to meet the demands.

A graphic comparison may be found in the production of Sulphuric Acid; long used as an index to the growth of the chemical industry. The vast "Lead chamber" plants with residence times of many hours could cover a hectare of land to produce a hundred tonne per day of 70% acid. "Contact" plants had taken over almost entirely by the 1960's, 300 tpd plants occupied no more than 1000 square metres and produced 98% acid and large quantities of process steam. By the 1970's a plant of double the capacity occupied a site half the size or smaller with faster reaction rates, shorter residence times and higher operating temperatures and pressures.

Eliminating the storage of intermediate products decreased the overall size of the plant but each unit now fed its product and its



disturbances directly into the unit downstream. An undetected or uncorrected disturbance could affect a whole section of plant to the detriment of product quality and operating economics. Improvements in the control of the processing environment may prevent some of the disturbances from reaching vital or unstable plant, or at least reduce the damaging effects, but many chemical processes remain very susceptible to product variability which cannot be entirely eliminated. Because of this the chemical manufacturer must frequently aim at a product specification exceeding that required by the purchaser, with consequent increase in costs or decrease in production rate, just to ensure that the natural variability of the process does not result in an unsaleable product.

Increased competition in the chemical market place led also to the general tightening of chemical and physical specifications encouraging tighter control of the manufacturing processes.

Recently two new factors have emerged which are having a very major influence on the chemical processing industries. The rapidly increasing cost and restricted supply of most forms of energy and, closely related, the increasing cost of petroleum, the basic feedstock for a considerable proportion of the chemical industry.

Throughout the last two decades the incentives for improvement in all aspects of process control could hardly have been greater - the same two decades which saw the most dramatic developments in the establishment of Modern Control Theory. How, then, was it possible for the new control theories and the established control practices to continue to develop and apparently thrive in virtual ignorance of each other's existence?

Turning to the current scientific literature we can view with awe the unending array of material being published on the subject of control science. The books and papers, however, all too often report what is theoretically possible, rather than what has been practically accomplished. The models and processes investigated appear to be designed to fit the control theories rather than to bear any relationship to the real world. The theory of modern control and the practice of control within the chemical processing technology appear to be poles apart and each must be studied in

relation to the other to understand how this may have come about. And to do that it is important to distinguish clearly between the modern control theories and the conventional approach to control system synthesis.

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### 1.1 SINGLE-LOOP CONTROL

The science of control grew from the single-input, single-output analysis of systems. Early "unit operations" equipment was generally operated at the level of sophistication which recognised a one-to-one relationship between the process variables and early automatic controllers implemented this philosophy. The process operator, as he learnt the idiosyncrasies of his plant, recognised the interaction effects and, if sufficiently skilled, used them to good effect. As interaction effects became more readily identified and quantified some were exploited in cascade controllers, but the basic principle of single-loop controllers remained. Even with the introduction of very obviously multi-input, multi-output processes and a greater understanding of the chemical and physical relationships involved the control systems remained essentially a collection of single-loop strategies.

One of the reasons for this must surely be the adherence to the laplace transfer function for process analysis which has until recently dominated the mathematical treatment of automatic control. Although both laplace domain and frequency domain techniques are adaptable to multi-input, multi-output systems there is something inherently clumsy about matrices of transfer functions. There is of course a direct correspondence between laplace and linear state-space presentation and all the operations in one domain have direct counterparts in the other.

In single-loop control each control variable is governed by the deviations of only one process variable. The deviations of other process

variables are ignored even though manipulation of that control variable would be beneficial in restoring those other variables to their respective set points. Neither are the interaction effects between the control variable and the other variables considered, except of course in the controller tuning once the plant is operating.

The control system's engineers were probably not chemical engineers and did not fully understand the chemical and physical phenomena they were attempting to control. It seems certain that the process designers did not set out to model the process with algebraic and dynamic equations for had they done so they could have hardly helped but notice the fundamental interactions between variables which are so obvious from the state-space representation.

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## 1.2 MULTI-VARIABLE CONTROL

Multivariable control recognises and uses the dynamic and algebraic relationships between variables. Each control variable is a function of all the process variables over which it has an influence. When any or all of the process variables deviate from their set-points the influence of the control variables on the system is to force the return of each variable to its set-point at a rate which is related to the penalty placed upon its deviation.

This raises another essential feature of Modern Control Theory, and that is the existence of a Process Control Criterion. The process controller is designed to optimise a process control criterion, however arbitrary that may appear to be, and involved in the choice of that criterion is the priority attached to the actions and interactions of the process variables. With the well-established Quadratic Error criterion, favoured for regulatory control, the squared deviations of the variables,

and sometimes their interaction products, are penalised according to an assessment of the detrimental effect of those deviations. The effect is such that the algorithm which optimises the control criterion is directed to reduce most rapidly the deviations of the most heavily penalised variables.

In a compact integrated plant the interactions between variables raises another major problem for single-loop control strategies - that of configuration. With so many alternatives to choose from how does the control engineer choose the best feed-back loops for process stabilisation? In a hypothetical, exothermic reactor does he use the deviation of the product concentration to control the feed rate, the supply of catalyst, the level of reactants in the reactor or the flow of coolant? Although intuition and Modern Control Theory say "use them all" the conventional control engineer must decide on just one, and make similar, perhaps arbitrary, decisions for the other loops as well. Cascading of control loops and the use of decoupling algorithms may go some way toward improving the quality of control, but the problems of the control engineer are only compounded by the increasing number and complexity of the possible candidates for the control loop configuration.

The dependence of the control variables on all the state variables means that a dynamic optimisation can be effected by the control function instead of the static optimisation performed by a single-loop control strategy. Returning to the exothermic reactor example; a change in feed concentration may be compensated most rapidly and economically by adjusting the reactor temperature. If, however, the coolant flow-rate is controlled only by the reactor temperature this dynamic compensation cannot be employed. The reactor temperature will eventually fluctuate because of the effect of the changed feed concentration on the reaction rate, but the response of the temperature controller will be delayed and the effect will be to try to restore the reactor temperature to the original set-point and not the temperature relevant to the current feed concentration. In effect, the action of the temperature controller will be contrary to the objective of maintaining a steady product concentration. Borer(1974) details the importance of dynamic compensation and emphasises the point (1977) with the comment that;

"If processes are designed with less and less holdup, higher

conversion rates, etc, then control must increasingly be exercised on a dynamic basis. Single loop regulation of individual process parameters is no longer adequate; disturbances cannot always be subdued before they enter the process and must be absorbed by permitting certain variables to change transiently in a controlled fashion."

If only some of the apparent advantages of multivariable control are able to be realised on a chemical plant it is still somewhat surprising to find such an obvious gap between the theory and the practice of modern control techniques. If, as is confidently predicted, process stability could be enhanced and product variability decreased by the use of multivariable controllers then why are control engineers and plant managers still wedded to the conventional control technology? The reasons are as numerous and as diverse as the chemical industry itself but a brief analysis of a few would be in order.

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### 1.3 CONVENTIONAL MULTI-LOOP TECHNOLOGY

"Better the devil you know ...."

The changes in chemical industry and the demands of new technology have, in general, been adequately met by improvements in conventional control. Operating speed and precision have increased and where necessary the slower and more cumbersome pneumatic systems have been supplanted by fast and dependable solid-state electronics, although even the standard pneumatic valve motors can be fitted with boosters and positioners which greatly improve their speed and accuracy. Complex and sophisticated single-loop algorithms can be implemented on a wide range of analogue and digital hardware and a range of mathematical functions are available. Control valves can be manufactured with programmed nonlinearities to compensate the processes they are controlling.

Above all this is the accumulation of experience which, for example, makes the configuration problem relatively tractable.

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#### 1.4 AUTOMATIC MEASUREMENT AND ANALYTICAL TECHNIQUES

Improvements in measurement technology have greatly assisted the control engineers in keeping up with modern processes, particularly the wide range of on-line automatic analysers. In the area of chemical analysis the range of automated techniques is extensive. Continuous monitoring of ionic concentration by specific ion electrodes is common, as is the discrete analysis by automatic titration. Ultraviolet and infrared spectrophotometry are also readily used on-line as are the very widely used chromatographic techniques. Even solids can be automatically analysed by methods such as X-ray diffraction and fluorescence. There are many techniques available for the on-line measurement of physical parameters; density, viscosity, refractive index, temperature, colour, turbidity, flow-rate, and conductivity may be readily determined for gases, liquids, and solids ranging from powders to larger particulates. The accuracy and speed of these analytical techniques are being steadily improved and many of the currently regarded off-line techniques may soon be fully automated for greater assistance with process optimisation and control.

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## 1.5 PROCESS CONTROL COMPUTERS

The introduction of digital computers to the chemical industry brought considerable advances to the science of control. As the digital computer is essentially a sampled-data machine it brought about something of a revolution in data acquisition and transmission, process representation and modelling, and the design of control algorithms. The speed of the digital computer meant that a much wider range of activities could be accommodated, including complex multi-loop control strategies and on-line optimisation.

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## 1.6 CHEMICAL PROCESS PLANT

If there is one factor which characterises chemical processes it would surely be non-linearity.

Non-Linearities abound in systems involved with heat, mass and momentum transfer, thermodynamics, reaction and catalysis and all the staged and recycle processes which render analytical techniques unworkable. Two other effects, endemic to chemical plant and similarly confounding the analytical approach, are hard constraints on process and control variables, and time delays.

Local linearisation is usually possible but the non-linearities may be so extreme that the region over which the linearisation may be applied is very small.

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## 1.7 MODERN CONTROL THEORY

"... than the devil you don't."

Modern control theory is unfortunately obscured and misrepresented by writers such as Foss (1973) in his "critique" of chemical process control theory and such reactionary attitudes are unfortunate. Certainly the problems associated with the design and application of Modern Control techniques are very far from any general solution, however the theoreticians have furnished us with a set of elegant theories, complete with necessary and sufficient conditions for global optimality. In contradiction to Foss, therefore, the ball is no longer in the theoreticians' court but very firmly established in the court of the control engineer.

For example: Local linearisation and judicious application of the quadratic error criterion result in a system for which a general solution has been established. That a model of the process to be controlled is necessary is not too great a hardship. If the process is in the design stages (surely the best time to be designing the control system) a mathematical model of the process is mandatory for good design. If the process is already in operation a modelling exercise could be of great benefit in understanding the process, and if an analytical approach is not appropriate then identification techniques could be employed. Hard constraints, time-delays and acute non-linearities are occasionally the result of poor design and as such may be revealed by a detailed analysis of the process. The cost of replacing a large diameter pipe with a small one and installing a pump to overcome the increased pressure drop, or replacing a control valve with one larger or faster, might be recovered in a matter of weeks by the reduced product variability due to faster and more positive control action.

Neither are the problems of high-dimensional models insurmountable. Those who cite the thousands of possible state and control variables within a large chemical complex as evidence of the impossibility of establishing a multivariable controller are either being naive or, like Foss, being misleading. Only the closely integrated sections of a plant need to be treated as units for the purposes of multivariable control and effective



model reduction techniques are available which will specifically retain the particular process dynamics that are regarded as important. Similarly the problem of inaccessible states may be accommodated by judicious model reduction or by the techniques of state and parameter estimation.

For the simplest case of time-invariant parameters and infinite horizon the implementation of a multivariable controller requires less control hardware than the equivalent multi-loop system with a separate controller for every loop. Workable algorithms are available from most modern texts on the subject of process control and could be applied by most engineers with a reasonable knowledge of linear algebra. Even discounting the analytical approach which is possible, if tedious, for systems of low order, the algorithms are readily programmed and solved off-line by a very modest computer. Contrary to popular belief a digital computer is not essential for the implementation of a multi-variable controller, although the existence of an on-line computer could improve the quality of control by means of non-linear controllers or with the techniques of state estimation, parameter tracking and process optimisation. The use of Modern Control techniques would require no more technical expertise than conventional control strategies.

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## 1.8 COMPARISON OF MULTIVARIABLE AND MULTI-LOOP CONTROL STRATEGIES.

Managers of chemical process plant, along with their process designers and control systems engineers, are essentially businessmen - hard-headed and perhaps a bit conservative, and therefore unlikely to abandon tried and proven methods unless a new system can prove its superiority in terms of processing economics. The stage has long since been reached in the application of Modern Control Theory that it needs to be rigorously tested against the well-established conventional techniques.

Conventional control technology has continued to survive under the conditions imposed by chemical processing; imprecise models, drifting parameters, measurement error, hard constraints and unavoidable dead-times. Unless the application of Modern Control Theory can consistently out-perform the established conventional control techniques the holders of the chemical industries' purse strings will continue to ignore it. And who could blame them?

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## CHAPTER II

## MODERN CONTROL TECHNIQUES

The State Variable form of process representation is introduced and some of the modern techniques for control system synthesis are examined.

	Page
2.1 State Representation of Dynamic Systems	18
2.2 Discrete-time State Equations	20
2.3 Eigenvalues and the System Response	21
2.4 Controllability and Observability	23
2.5 The Development of Single-Loop Control Laws	24
2.6 Modern Control Techniques	25
2.7 Direct Search Techniques	26
2.8 Modal Control	27
2.9 Variational Methods of Optimisation	29
2.10 The Calculus of Variations	29
2.11 Poultryagin's Minimum Principle	31

## CHAPTER II

## MODERN CONTROL TECHNIQUES

## 2.1 STATE REPRESENTATION OF DYNAMIC SYSTEMS.

In the development and discussion of various techniques for constructing multivariable control algorithms the dynamic systems will be represented exclusively in State Variable form. The techniques for conversion between linear state variable and laplace forms are well summarised by Ward and Strum (1970) and Noton (1972). The development of mathematical models of chemical processes in state variable form is covered in many modern texts and the application to particular systems will be described in some detail in later chapters. The most general form of state variable representation of a process is

$$\dot{x}_i = f_i(x, u, d, t) \quad \text{.....(2.1)}$$

$$i = 1, 2, \dots, n$$

where  $x$  is the state vector,  $u$  is a vector of control variables and  $d$  is a vector of disturbance variables. Where the process is linear, or may reasonably be linearised about a point in state space, the process may be presented in the standard non-homogeneous linear form

$$\dot{x} = A(t)x(t) + B(t)u(t) + C(t)d(t) \quad \text{.....(2.2)}$$

where  $A$  is the State matrix and  $B$  and  $C$  are driving matrices relating to the control inputs,  $u$ , and the process disturbance inputs,  $d$ , respectively. The driving matrices are sometimes combined into a single matrix and a combined vector is formed from the control and disturbance vectors. This is the most compact linear form but it tends to obscure the important distinction between those inputs which may be manipulated to optimise the process and the uncontrolled disturbance, or "load", variables.

Despite the dexterity with which analogue and digital computers can generate solutions (or trajectories) for sets of non-linear differential equations there are considerable advantages in the use of linear equations in terms of speed, generality and the existence of analytical solutions.

For non-linear systems the effort involved in determining truly optimal control algorithms is very considerable and generally prohibitive for on-line work. Conversely, local linearisation and the development and application of linear controllers is generally more straightforward and should appeal to engineering pragmatism.

For a process modelled by a set of non-linear equations

$$\dot{X} = f(X, U, t) \quad \text{.....(2.3)}$$

it is possible to linearise about a point  $(\bar{X}, \bar{U})$ , or perhaps a nominal trajectory  $(\bar{X}(t), \bar{U}(t))$ , by defining deviation variables  $x, u$  such that

$$x = X - \bar{X}, \quad u = U - \bar{U}. \quad \text{The model may be expressed in terms of these new variables} \quad x = f(x, u, \bar{X}, \bar{U}, t) \quad \text{.....(2.4)}$$

and may then be linearised about a point or nominal trajectory, by calculating the Jacobian matrices  $J_x$  and  $J_u$  whose elements are given by

$$(J_x)_{ij} = \frac{\partial f_i}{\partial x_j} \quad \text{and} \quad (J_u)_{ij} = \frac{\partial f_i}{\partial u_j} \quad \text{.....(2.5)}$$

and are evaluated at the point or along the trajectory about which the model was to be linearised. The Jacobean matrices  $J_x$  and  $J_u$  are the state and driving matrices respectively.

The general solution to sets of linear equations of the form

$$\dot{x} = A(t)x(t) + B(t)u(t) \quad \text{.....(2.6)}$$

may be expressed as

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^t \phi(t, \tau)B(\tau)u(\tau)d\tau \quad \text{.....(2.7)}$$

where  $\phi(t, t_0)$  is known as the Transition matrix and is defined as the solution to the homogeneous matrix differential equation

$$\frac{d}{dt} \phi(t, t_0) = A(t)\phi(t, t_0) \quad \text{.....(2.8)}$$

for all  $t$ . The solution to this equation for a time invariant state matrix  $A$  takes the form of a matrix exponential

$$\phi(t, t_0) = \exp(A(t-t_0)) \quad \text{.....(2.9)}$$

which may be expressed and evaluated as the matrix series

$$\begin{aligned} \exp(A(t-t_0)) &= I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \dots \\ &\dots + \frac{A^r(t-t_0)^r}{r!} + \dots \quad \text{.....(2.10)} \end{aligned}$$

and which can be shown to converge absolutely and uniformly on any finite interval of the time axis, (Tou, 1964), and is readily programmed for solution by digital computer (Noton, 1972, Kalman and Englar, 1966, Melsa, 1970). The proof of the existence and uniqueness of the matrix series solution for the general case, and its non-singularity for all  $(t, t_0)$ , is given by Bellman (1967). There are problems with truncation and rate of convergence however when the matrix  $A(t-t_0)$  is large and the time interval  $(t-t_0)$  may have to be reduced in order to effect a satisfactory numerical solution. The solution to the non-homogeneous part would generally be calculated at the same time provided the characteristics of the forcing function,  $u$ , are known over the range  $(t, t_0)$ . The complete solution, comprising the sum of the free and forced responses, may be expressed in terms of the transition and driving matrices

$$x(t) = \Phi(t, t_0)x(t_0) + \Delta(t, t_0)u(t) \quad \dots\dots(2.11)$$

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## 2.2 DISCRETE-TIME STATE EQUATIONS.

The application of digital computers to process control requires the development of discrete-time models. The digital control computer is a sampled-data device and the computations are handled in a stage-wise fashion. A process model must therefore be devised to provide information on the state variable trajectory at discrete instants corresponding to the control computer's view of the continuous process. This is done by solving the set of continuous differential equations over the discrete time interval  $T$  assuming piecewise continuous inputs for the duration of the interval. The solution, once again straightforward for sets of linear equations, involves the evaluation of the transition and driving matrices

$$\begin{aligned} \Phi(t, t_0) &= \Phi(T) \\ \Delta(t, t_0) &= \Delta(T) \end{aligned} \quad \dots\dots(2.12)$$

The state of the process at the instant of sampling may thus be

expressed as the sum of a transformation of the state at the previous sampling instant and the effect of the process inputs for the duration of that stage, i.e.

$$x((k+1)T) = \Phi(kT)x(kT) + \Delta(kT)u(kT) \quad \text{.....(2.13)}$$

This may be expressed more conveniently as

$$x(k+1) = \Phi(k)x(k) + \Delta(k)u(k) \quad \text{where the sampling interval is implied, or even}$$

$$x_{k+1} = \Phi_k x_k + \Delta_k u_k \quad \text{.....(2.14)}$$

The more compact subscripted form of equation 2.14 is preferred where it is clear from the context that the subscript refers to the stage and not to a specific element in the vector.

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### 2.3 EIGENVALUES AND THE SYSTEM RESPONSE.

The choice of the state variables is generally straightforward for chemical processing systems, using either basic or derived measurements. The state of a sucrose solution in an evaporator might be described in terms of Temperature and Concentration but for the purposes of on-line analysis the state may be more conveniently measured in terms of its density and refractive index. Temperature and concentration are however independent modes in that either may change (within limits) without affecting the other, whereas a change in density, for example, brought about by a change in either temperature or concentration, would also have an effect on the refractive index. There are a number of physico-chemical properties of the sucrose solution which could be measured, indicating that the choice of state variables is not unique, but there is one set of state variables, the dynamic behaviour of which displays the system structure in its most basic form because of their lack of interaction with one another. These modes are linear combinations of the measured properties and are known as eigenvectors, and the transformation of the states into the respective eigenvectors results in a state matrix in diagonal canonical form.



For a linear dynamic process

$\dot{x} = Ax + Bu$  let the linear transformation which achieves this decoupling of the state variables be  $P$ , such that

$$\begin{aligned} x &= Pz \\ \text{then } \dot{z} &= P^{-1} \dot{x} \\ &= P^{-1} (APz + Bu) \\ &= P^{-1} APz + P^{-1} Bu \end{aligned} \quad \text{.....(2.15)}$$

and the state matrix which makes the transformed state variables independent is the diagonal matrix of eigenvalues, or Spectral Matrix,  $\Lambda$ .

$$\begin{aligned} P^{-1}AP &= \Lambda \\ &= \text{diag}(\lambda_1 \lambda_2 \dots \lambda_n) \end{aligned} \quad \text{.....(2.16)}$$

The free time-response of the modes is

$$z_i(t) = \exp(\lambda_i t) z_i(t_0) \quad \text{.....(2.17)}$$

and the mode will be stable if the real part of the eigenvalue  $\lambda_i$ , is negative. For complex eigenvalues the system response will be oscillatory.

For discrete time systems the free response is given by

$$x(k+1) = \Phi x(k) \quad \text{.....(2.18)}$$

and the process is stable if the sequence of euclidean norms of the state vector

$$\|x(k)\| = \left[ \sum_{i=1}^n x_i(k)^2 \right]^{1/2}$$

is a decreasing monotonic sequence.

For this to be so the necessary and sufficient condition is  $\|\Phi\| < 1.0$  implying that the eigenvalues of the matrix  $\Phi$  lie within the unit circle on the complex plane.

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## 2.4 CONTROLLABILITY AND OBSERVABILITY.

In general terms a system is controllable if any desired change of the system state can be achieved in a finite time by the application of unconstrained control action.

A system is observable if any change to the system will eventually have an effect on the system output.

The development of the concepts of controllability and observability is contained in the pioneering papers of Kalman (1960, 1961, 1963). For a detailed discussion of the dual properties of controllability and observability (or reconstructability) the interested reader should refer to Lapidus & Luus (1967, p35) or Kwakernaak and Sivan (1972, p53).

Controllability is usually assessed by ensuring that the composite  $n \times mn$  matrix

$[B:AB:A^2B:\dots:A^{n-1}B]$  is of rank  $n$ . This would be equivalent to ensuring that

$[\Delta:\phi\Delta:\phi^2\Delta:\dots:\phi^{n-1}\Delta]$  is of rank  $n$  for the discrete time model. The concept of controllability may be more readily understood, however, by examining the state equations in diagonal canonical form, i.e.

$$\dot{z} = \Lambda z + P^{-1}Bu \quad \dots\dots(2.19)$$

In this form it is easy to see that the modes of the system have no influence on each other, being mutually orthogonal, and the response of each mode may be affected only by the controls  $u$ . If, however, any row of the matrix  $P^{-1}B$  comprises only zeros control action cannot influence the behaviour of that mode and it is "not controllable".

In many chemical processes the complete state vector  $x_i, i = 1, 2, \dots, n$ , will not be available for identification or control purposes and an output vector  $y = Hx$  is defined, where  $y$  is a vector of  $l$  components and the  $l \times n$  measurement matrix  $H$  must be of rank  $l$  to ensure  $l$  independent measurements. If the  $n \times ln$  matrix

$[H^T:A^T H^T:(A^T)^2 H^T:\dots:(A^T)^{n-1} H^T]$  is of rank  $n$  the

system is observable, meaning that it is possible to derive a sufficient number of linearly independent equations to solve for the  $n-1$  states which do not appear in the output vector. The observability test is of limited value as a means of state estimation since the complete state at some time  $t$  may be determined only by observing the output over a period of time subsequent to  $t$ . Also the states may well be subject to random measurement errors which would probably render the result meaningless. The test is of importance however when methods of state estimation and prediction are being investigated by means of an observer such as proposed by Luenberger (1966) or, in the case of noisy measurements, the predictive filter of Kalman (1960).

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## 2.5 THE DEVELOPMENT OF SINGLE-LOOP CONTROL LAWS.

Chemical processes have traditionally been controlled by continuous or discrete controllers using combinations of Proportional, Integral and Derivative modes. With the conventional control system a control variable is manipulated, by means of a controller, according to an input signal which, in the case of a feedback controller, would be the deviation of a selected process variable from its required operating level, or set-point. The techniques for determining suitable numerical values for the proportional gain and the integral and derivative time constants have generally involved the dynamics of only that process variable. The extent to which the controller performs a genuine optimising function depends on the basis on which the controller parameters were chosen, but most techniques may be resolved into a relatively arbitrary balance between maximising the speed of response, minimising the peak deviation, the offset and the degree of oscillation, and ensuring stability in the face of random disturbances over as wide a frequency spectrum as possible. The single-loop controller receives signals from only one process variable and is unable, therefore, to perform anything other than a very localised optimisation.

The techniques for process characterisation and the subsequent calculation of controller parameters may be found in many texts and papers but attention should be drawn to the pioneering paper by Ziegler and Nichols (1942). It is significant that for a process which can be characterised as a first order stage plus dead-time the Ziegler - Nichols settings are still regarded as the best initial estimate. Cohen and Coon (1953), also of the Taylor Instrument Companies, in a theoretical investigation of the same process model using frequency response techniques, produced controller tuning charts based on stability criteria.

For a general study of the application of conventional control techniques to chemical processes the basic texts would have to include Buckley (1964), Gould (1969), Luyben (1969), Shinskey (1967), Harriott (1964) and Coughanowr and Koppel (1965).

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## 2.6 MODERN CONTROL TECHNIQUES

"Optimal" control, as the name suggests, involves the optimisation of a functional which represents a numerical criterion of the system's performance. The choice of criterion may be as arbitrary as the commonly used 4:1 damping ratio of conventional control, but on the other hand it may be related very specifically to the economics of the processing operation. The important aspect to recognise is that the optimal control law emerges as the solution to the optimisation exercise.

The control law may be formulated in two distinct ways; the control vector may be stated as a function of the process state, in which case it is a closed-loop or "feed-back" law, or it may be expressed as a function of the initial conditions and time and is therefore an open-loop law. Regulatory control is almost always of closed-loop form which has two distinct advantages:

(a) The process does not require a particularly accurate model (although the better the model, of course, the better the control) and

(b) The control sequence is not invalidated by extraneous disturbances.

Open-Loop, or programmed, control is occasionally employed for batchwise processing or plant start-up sequencing but frequently employs some feed-back mechanism if the end-point is at all critical.

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## 2.7 DIRECT SEARCH TECHNIQUES.

Optimal control is frequently regarded as synonymous with Multivariable control since the solution to the control criterion optimisation results in a multivariable control law. It is possible, however, to construct a multivariable controller by a straightforward, if somewhat tedious, search for the appropriate elements of the feedback matrix  $K_{FB}$ . It is most unlikely that this approach would be attempted on-line so an adequate model would be essential. The search would be prohibitive for large systems, even with a very efficient optimisation routine, since for a process with  $n$  states and  $m$  controls the number of independent coefficients is  $n*m$ . Each simulated run would need to be of sufficient duration to measure accurately the performance of each candidate controller.

Where a reasonably accurate initial estimate is available the method may be tractable and could result in a viable technique for approximating to the optimal feedback matrix for a small system with some inaccessible states.

The direct search technique has been used by Luus(1974) for three non-linear chemical engineering systems and constant feedback matrices have been found to minimise quadratic criteria and the time to reach the origin.

The method involves the selection of an initial feedback matrix and a range about which each element in the matrix may be perturbed. The feedback matrix is then perturbed by sets of random numbers in the interval  $(-0.5, 0.5)$  and each resultant matrix is tested for optimality. The range is then reduced slightly and the process repeated until eventually stationarity occurs. The method is simple to program and results in a useful feedback controller, but only one set of initial conditions for the non-linear process were used which results in a gain matrix specific to those initial conditions. Also the time spans were very short which would have reduced the computation times. Luus did not encounter any problems with multiple solutions as he had with the same non-linear systems using a gradient search technique (1973), the result, perhaps, of informed initial policies and large ranges. It would be prudent however to test the solution by using a very different initial feedback matrix.

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## 2.8 MODAL CONTROL

Another technique for generating multivariable controllers which has received considerable attention in the literature is pole-assignment, or modal control. Since modal control does not involve the optimisation of a process criterion, however, it is not strictly an optimal control technique. The principle is to shift the system eigenvalues by means of the feedback matrix but the theory does not extend to providing guidance on where to shift the eigenvalues in order to obtain a specified response. Since the transient behaviour of the system is predominantly determined by the mode associated with the slowest eigenvalue, the technique has resolved into shifting this eigenvalue as far to the left on the complex plane as the control variable amplitudes permit, ensuring stability and improving the speed of response. Rosenbrock (1962) considered the technique appropriate to chemical engineering processes and proposed the conversion to diagonal closed-loop form. For the standard process model

$$\dot{x} = Ax + Bu \quad \text{with closed-loop control law}$$

$u = Kx$  The system may be transformed into mode-space by the transformation

$$\begin{aligned} x &= Vz \quad \text{where } V \text{ is the modal matrix of } A \\ \dot{x} &= V\dot{z} \\ &= AVz + Bu \end{aligned}$$

$$\text{whence} \quad \dot{z} = V^{-1}AVz + V^{-1}Bu \quad \dots\dots(2.20)$$

but since  $V^{-1}AV = \Lambda$ , the spectral matrix, and with

$$\begin{aligned} u &= KVz, \text{ the closed-loop form becomes} \\ \dot{z} &= (\Lambda + V^{-1}BKV)z \quad \dots\dots(2.21) \end{aligned}$$

and proper selection of the matrix  $K$  results in the desired closed-loop response. Gould (1960) provides a concise derivation of the method. Ellis and White (1965) present a detailed development of single-loop modal control and its application to boiler pressure control, while Porter and Micklethwaite (1967) describe the procedures for both sequential and simultaneous modal controller design, demonstrating that for a desired set of eigenvalues the solution is not necessarily unique. They apply the method to a set of interacting tanks but the results show that the required control variable amplitudes are quite unrealistic.

Davison and Goldberg (1969) extend the work of Rosenbrock (1962) with a technique for the elimination of inaccessible states and this method is applied by Davison and Chatha (1972) with some success. The most definitive study is probably the text by Porter and Crossley (1972).

Bruun (1975) describes in detail the application of modal control to an experimental evaporator but found the resultant controller to have poor load rejection properties and steady-state offsets. The addition of integral states, and thus zero eigenvalues, was found to reduce the offsets but Bruun concluded that modal control had considerable deficiencies compared to optimal control techniques.

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## 2.9 VARIATIONAL METHODS OF OPTIMISATION.

The methods for direct optimisation of the process criterion are generally classified as Variational techniques and they include the classical Calculus of Variations, the superficially similar Minimum Principle of Pontryagin and the superficially different approach of Dynamic Programming.

## 2.10 THE CALCULUS OF VARIATIONS

The problem may be stated as the optimisation of a functional subject to the constraints imposed by the system in the form of a set of dynamic state equations. The most general criterion consists of a functional of the final state of the process combined with a functional derived from the total process trajectory (a combination of the problems of Mayer and Bolza).

$$J(u) = F[x(t_f)] + \int_{t_0}^{t_f} L[x(t), u(t), t] dt \quad \dots\dots\dots(2.22)$$

and the minimisation of equation 2.22 with respect to the control vector  $u(t)$  is subject to the constraints

$$\dot{x}_i = \varphi_i(x, u, t) \quad \dots\dots\dots(2.23)$$

$$i = 1, 2, \dots, n$$

The constraints are included by means of an adjoint vector which consists of dynamic terms, usually called the co-state variables.

The optimisation of the functional

$$J(u) = F[x(t_f)] + \int_{t_0}^{t_f} L(x, u, t) + \lambda^T(\varphi(x, u, t) - \dot{x}) dt \quad \dots\dots\dots(2.24)$$

is sought. The scalar Hamiltonian may be defined as

$$H = L(x, u, t) + \lambda^T \varphi(x, u, t) \quad \dots\dots\dots(2.25)$$

and equation 2.24 becomes



$$\begin{aligned}
J(u) &= F[x(t_f)] + \int_{t_0}^{t_f} (H - \lambda^T \dot{x}) dt \\
&= [F(x) - \lambda^T x]_{t_f} + [\lambda^T x]_{t_0} + \int_{t_0}^{t_f} [H + \lambda^T x] dt \quad \dots\dots\dots(2.26)
\end{aligned}$$

The first variation of the integral is

$$\delta J = [(F_x - \lambda^T) \delta x]_{t_f} + [\lambda^T \delta x]_{t_0} + \int_{t_0}^{t_f} [H_u \delta u + H_x^T \delta x + \lambda^T \delta x] dt \quad \dots\dots\dots(2.27)$$

where  $H_x = \partial H / \partial x$ , etc. In order that  $\delta J = 0$  for all small but arbitrary variations of  $\delta x$ ,  $\delta u$  the integral must be zero throughout the interval  $(t_0, t_f)$ , i.e.

$$\frac{\partial H}{\partial u} = 0 \quad \dots\dots\dots(2.28)$$

$$\frac{\partial H}{\partial x} + \dot{\lambda} = 0 \quad \dots\dots\dots(2.29)$$

Equation 2.29 defines the  $n$  components of the adjoint vector  $\lambda(t)$  by the set of differential equations

$$\dot{\lambda}^T(t) = - \frac{\partial H^T}{\partial x} = - \frac{\partial H}{\partial x} - \lambda^T \frac{\partial \varphi}{\partial x} \quad \dots\dots\dots(2.30)$$

The first term of equation 2.26 provides the boundary conditions

$$\lambda(t_f) = \frac{\partial F}{\partial x} \Big|_{t_f} \quad \dots\dots\dots(2.31)$$

since perturbations of the initial conditions may be ignored.

Equations 2.28 and 2.31 constitute the necessary conditions for a stationary solution. For the development of additional necessary and sufficient conditions for a global minimisation the reader is referred to Bryson and Ho (1969) and a summary only will be included here.

The Legendre-Clebsch Condition referred to in the classical literature is the necessary condition

$$\begin{aligned}
H_{uu} &\geq 0 \quad \text{which becomes a sufficient condition for a} \\
&\text{local minimum (i.e. the Convexity Condition) if} \\
H_{uu} &> 0
\end{aligned}$$

The Weierstrass Condition provides a strong necessary condition for a local minimum if

$$H(x^0, \lambda^0, u, t) - H(x^0, \lambda^0, u^0, t) \geq 0 \quad \text{.....(2.32)}$$

for all  $t$ , and  $u \neq u^0$ .

The Normality Condition demands that when terminal constraints are included the system be controllable throughout the interval  $(t_0, t_f)$ .

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## 2.11 PONTRYAGIN'S MINIMUM PRINCIPLE.

Pontryagin's Principle will be stated without proof; interested readers should consult the english translation of the original text by Pontryagin, et al, (1962).

Given the same state equations and criterion as previously (equations 2.22, 2.23) the co-state variables are defined as

$$\dot{\lambda}(t) = - \frac{\partial H}{\partial x} \quad \text{.....(2.33)}$$

with boundary conditions provided by

$$\lambda(t_f) = \left[ \frac{\partial F}{\partial x} \right]_{t_f} \quad \text{.....(2.34)}$$

the Principle may now be stated as:

"In order to minimise the performance functional (equation 2.22) the Hamiltonian  $H$  must be minimised at all times over all possible values of the control vector  $u$ ."

The mathematical description shows that it conforms to the Weierstrass Condition providing a strong necessary condition for a local minimum.

$$H(x^0, \lambda^0, u^0, t) \leq H(x^0, \lambda^0, u, t) \quad \text{.....(2.35)}$$

for all  $t$ , and  $u \neq u^0$ .

An important point to note is that inequality constraints on the control variable may be handled conveniently without any of the

mathematical artifices required by the Calculus of Variations. Previously the optimality condition has been

$\partial H / \partial u = 0$  appropriate to unconstrained control variables, whereas Pontryagin's Principle allows the selection of a least value of the Hamiltonian whether or not it occurs at a stationary point. Other advantages which make Pontryagin's Principle more powerful as well as more convenient are that the state equations are no longer required to have continuous first partial derivatives and, since the Weierstrass Condition is a strong necessary condition for optimality, the variations  $\delta u(t)$  need not be small.

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## CHAPTER III

### DYNAMIC PROGRAMMING

Dynamic Programming is introduced and developed as a process control technique for both discrete and continuous systems. The relationship between Dynamic Programming and the Variational Calculus is demonstrated, and techniques for solving the matrix Riccati equation are presented.

	Page
3.1 Introduction	34
3.2 Control of a Linear First-Order Process	36
3.3 Dynamic Programming and Multivariable Discrete Systems	38
3.4 The Application of Dynamic Programming to Continuous Processes	40
3.5 Dynamic Programming and the Calculus of Variations	42
3.6 Analytical Solution of the Hamilton-Jacobi Equation	44
3.7 Newton-Raphson Solution of the Steady-State Riccati Equation	46
3.8 Direct Euler Integration of the Matrix Riccati Equation	47
3.9 The Kalman-Englar Technique	47

## CHAPTER III

## DYNAMIC PROGRAMMING

## 3.1 INTRODUCTION.

The dynamic programming approach to a variational problem appears very different to that of the Calculus of Variations. Instead of determining the entire extremal trajectory the dynamic programming approach is to develop the extremal curve by evaluating the optimal derivative at each point, thus constructing an envelope of tangents. Where the Calculus of Variations may be described as a global approach, dynamic programming is essentially a local approach.

The theory of dynamic programming was developed by R.E. Bellman at the RAND Corporation in 1949 and the first formal expository paper (Bellman, 1952) was published by the National Academy of Science, U.S.A.

The intimate connection between dynamic programming and Modern Control Theory was outlined by Bellman (1961) and the application to control problems, as well as a very detailed bibliography, has also been provided by Bellman (1967, 1971). For an introduction to dynamic programming as a general optimisation technique the interested reader should refer to the texts by Roberts (1964), Lapidus and Luus (1967), Beveridge and Schechter (1970), Boudarel, et al, (1971) and Noton (1972).

Dynamic programming is based upon the Principle of Optimality which may be stated as follows:

"an optimal policy has the property that whatever the initial state and initial decision the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."

This apparent truism leads to the recurrence equation for multi-stage optimisation which is the basis of the dynamic programming technique. The development of this recurrence relationship has been demonstrated by Bellman

(1961,1967), Roberts (1964), Tou (1964), Lapidus and Luus (1967), Boudarel, et al, (1971) and Noton (1972). Although there is little merit in reproducing it in detail the technique may be demonstrated by application to a simple stagewise process.

Before proceeding, however, it is necessary to define the terms to be used throughout this section. The duration of the process is divided into "stages" corresponding to the discrete time interval and the stages are numbered in the direction of increasing time starting from stage zero. This has the minor disadvantage that the final stage of an N-stage process is numbered N-1, but the advantages are substantial.

The "state" of the process is the collection of variables, expressed as a vector  $x$ , necessary to describe the condition of the process. Where the state of the process may be observed only at the discrete instants between stages the vector  $x_k$  is defined as the state of the process at the beginning of stage  $k$ . The process thus has initial and final states  $x_0$  and  $x_N$  respectively and the process may be modelled by the linear equation

$$x_{k+1} = \Phi_k x_k + \Delta_k u_k \quad \dots\dots(3.1)$$

where  $\Phi_k$  and  $\Delta_k$  are the transition and driving matrices appropriate to stage  $k$  and  $u_k$  is the control policy used throughout stage  $k$ .

In general terms the control criterion  $J(x, u, N-k)$  may be regarded as the cost incurred in traversing from stage  $k$ , state  $x_k$ , to the end of the control sequence using a control policy  $u$ . If  $u$  is chosen to be an optimal policy the optimal criterion  $J^0$  will be given by

$$J^0(x_k, N-k) = \max_{u_i} [J(x_k, u, N-k)] \quad \dots\dots(3.2)$$

$i = k, k+1, \dots\dots, N.$

where  $N$  is the total number of stages in the sequence. If the cost incurred in traversing just the  $k$ 'th stage is  $L(x_k, u_k, k)$ , then the optimal criterion may be expressed as

$$J^0(x_k, N-k) = \max_{u_k} [L(x_k, u_k, k) + J^0(x_{k+1}, N-k-1)] \quad \dots\dots(3.3)$$

since the principle of optimality states that the policy from stage  $k+1$  onwards must constitute an optimal policy using  $x_{k+1}$  as its initial state.

This is the recurrence relationship and it will be seen that the multi-parameter optimisation of equation 3.2 has been reduced to a single-parameter optimisation in equation 3.3 by imbedding the optimal solution for all the subsequent stages within the optimisation of stage  $k$ . The sequence may be initiated by establishing the terminal cost  $J^0(x_N, 0)$  by calculus or iterative techniques. An important feature of the recurrence relationship is that the optimal control policy is developed as a feedback law and this aspect will be demonstrated by the application to a linear first-order system.

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### 3.2 CONTROL OF A LINEAR FIRST-ORDER PROCESS.

For the process described by the equation

$$x_{k+1} = ax_k + bu_k \quad \text{.....(3.4)}$$

the quadratic criterion

$$J(x_0, u, N) = \sum_{k=0}^N [x_k^2 + ru_k^2] \quad \text{.....(3.5)}$$

is to be minimised and the minimising sequence of controls results in the optimal criterion

$$J^0(x_0, N) = \min_{u_i} [J(x_0, u, N)] \quad \text{.....(3.6)}$$

$$i = 0, 1, \dots, N-1.$$

The cost for stage  $k$  is

$$L(x_k, u_k) = x_k^2 + ru_k^2 \quad \text{.....(3.7)}$$

and the optimal criterion at the  $k$ 'th stage may be expressed in recurrence equation form

$$\begin{aligned} J^0(x_k, N-k) &= \min_{u_k} [L(x_k, u_k) + J^0(x_{k+1}, N-k-1)] \\ &= \min_{u_k} [x_k^2 + ru_k^2 + J^0(x_{k+1}, N-k-1)] \quad \text{.....(3.8)} \end{aligned}$$

The sequence is initialised by establishing the terminal criterion

$$J^0(x_N, 0) = \min_{u_N} [x_N^2 + ru_N^2]$$

$$= x_N^2 \quad \dots\dots\dots(3.9)$$

with optimising control variable  $u_N = 0.0$

The optimal terminal criterion may now be imbedded into the recurrence equation to solve for the final stage.

$$\begin{aligned} J^0(x_{N-1}, 1) &= \min_{u_{N-1}} [J(x_{N-1}, u_{N-1}, 1)] \\ &= \min_{u_{N-1}} [L(x_{N-1}, u_{N-1}) + J^0(x_N, 0)] \\ &= \min_{u_{N-1}} [x_{N-1}^2 + ru_{N-1}^2 + x_N^2] \end{aligned}$$

but from equation 3.4

$$x_N = ax_{N-1} + bu_{N-1} \quad \text{therefore}$$

$$J^0(x_{N-1}, 1) = \min_{u_{N-1}} [x_{N-1}^2 + ru_{N-1}^2 + (ax_{N-1} + bu_{N-1})^2] \quad \dots\dots\dots(3.10)$$

which is analytic in  $u_{N-1}$  and the minimising control is found from

$$\partial J / \partial u_{N-1} = 0 \quad \dots\dots\dots(3.11)$$

giving

$$u_{N-1} = -(ba/(r+b^2))x_{N-1} \quad \dots\dots\dots(3.12)$$

and confirming the negative feedback nature of the control law. The sequence is continued toward the origin with the optimising control generated as a sequence of feedback gains, each appropriate to its stage but independent of the state at that stage.

The combination of linear process and quadratic criterion produces an optimal criterion of quadratic form, e.g.

$$J^0(x_k, N-k) = g_k x_k^2 \quad \dots\dots\dots(3.13)$$

and substitution of this into equation 3.10 gives

$$g_k x_k^2 = \min_{u_k} [x_k^2 + ru_k^2 + g_{k+1} (ax_k + bu_k)^2] \quad \dots\dots\dots(3.14)$$

and the right hand side is minimised for

$$u_k = -d_k x_k$$

where

$$d_k = (abg_{k+1} / (r + b^2 g_{k+1})) \quad \dots\dots\dots(3.15)$$

and substitution of the minimising value of  $u_k$  into equation 3.14 gives



$$[q_k - (1 + rd_k^2) - (a - bd_k)^2 q_{k+1}] x_k^2 = 0. \quad \text{.....(3.16)}$$

which may be satisfied for all  $x_k$  only if

$$q_k = 1 + rd_k^2 + (a - bd_k)^2 q_{k+1} \quad \text{.....(3.17)}$$

Equations 3.15 and 3.17 define the iterative process for determining the sequence of feedback gains and optimal criteria. An important aspect, demonstrated later, is that the optimal criterion and the feedback gain converge to steady-state values as  $N$  increases leading to a constant feedback gain for a process with an unbounded time horizon.

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### 3.3 DYNAMIC PROGRAMMING AND MULTIVARIABLE DISCRETE SYSTEMS.

For a multivariable process which may be modelled in the linear discrete form

$$x_{k+1} = \phi_k x_k + \Delta_k u_k \quad \text{.....(3.18)}$$

and with quadratic criterion

$$J = x_N^T S x_N + \sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) \quad \text{.....(3.19)}$$

a similar iterative technique may be established to generate the sequence of linear multivariable feedback matrices. Because of the linear dependence of the series  $x_k$  and  $u_k$  the optimal criterion  $J^0$  will be strictly convex in  $u$  and a unique minimum exists provided the symmetric matrices  $S$ ,  $Q$  and  $R$  are positive semi-definite. As in the single-variable example the optimal criterion may be assumed to have the quadratic form

$$J^0(x_k) = x_k^T G_k x_k \quad \text{.....(3.20)}$$

and this may be substituted into the recurrence equation 3.8, using the process model 3.18 to eliminate the term in  $x_{k+1}$ , to give

$$x_k^T G_k x_k = \min_{u_k} [x_k^T Q_k x_k + u_k^T R_k u_k + (\phi_k x_k + \Delta_k u_k)^T G_{k+1} (\phi_k x_k + \Delta_k u_k)] \quad \text{.....(3.21)}$$

The minimisation of the right hand side of equation 3.21 is accomplished by partial differentiation with respect to each of the  $m$  elements of the control vector  $u_k$  and minimising values are to be found from the linear relationship

$$u_k = -K_k^T \phi_k x_k \quad \text{.....(3.22)}$$

where the matrix  $K_k$  is obtained from

$$K_k = G_{k+1} \Delta_k [R_k + \Delta_k^T G_{k+1} \Delta_k]^{-1} \quad \text{.....(3.23)}$$

Substitution of the optimising value of the control vector into equation 3.21 provides the optimal criterion

$$J^0(x_k) = x_k^T G_k x_k \quad \text{which may be rearranged to give}$$

$$x_k^T [Q_k - G_k + \phi_k^T P_k \phi_k] x_k = 0 \quad \text{.....(3.24)}$$

where  $P_k = G_{k+1} - K_k \Delta_k^T G_{k+1} \quad \text{.....(3.25)}$

Since the optimal criterion must be valid for all  $x_k$  equation 3.24 may be rearranged to give

$$G_k = \phi_k P_k \phi_k + Q_k \quad \text{.....(3.26)}$$

and equations 3.22, 3.23, 3.25 and 3.26 define the iterative process from which the sequence of control vectors  $u_k$  may be obtained.

With the criterion as defined by equation 3.18 the sequence would be initiated by  $G_N = S$ . Where the steady-state feedback matrix is required, however, the initial matrix is of little significance and has no effect on the steady-state solution. Sufficiency conditions for the convergence of the matrix sequence have been presented by Caines and Mayne(1970).

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### 3.4 THE APPLICATION OF DYNAMIC PROGRAMMING TO CONTINUOUS PROCESSES.

For a continuous process described by the set of equations

$$\dot{x} = \varphi(x, u, t) \quad \dots\dots(3.27)$$

the process criterion may be expressed as the functional

$$J = \int_{t_0}^{t_f} L(x, u, t) dt \quad \dots\dots(3.28)$$

The optimal criterion  $J^0$ , at arbitrary time  $t$  and state  $x$ , may be found by the minimisation of the criterion with respect to the control vector  $u$ . From the Principle of Optimality the trajectory to the final time  $t_f$  must be optimal and the criterion will be a function of both  $x$  and  $t$ , thus

$$\begin{aligned} J^0(x, t) &= f(x, t) \\ &= \min_u \int_t^{t_f} L(x, u, t) dt \quad \dots\dots(3.29) \end{aligned}$$

If the process is optimised over two stages,  $(t, t+\delta)$  and  $(t+\delta, t_f)$ , the optimal criterion becomes

$$\begin{aligned} f(x, t) &= \min_u \left[ \int_t^{t+\delta} L(x, u, t) dt + \int_{t+\delta}^{t_f} L(x, u, t) dt \right] \\ &= \min_u \left[ \int_t^{t+\delta} L(x, u, t) dt + f(x+\dot{x}\delta, t+\delta) \right] \quad \dots\dots(3.30) \end{aligned}$$

where

$$f(x+\dot{x}\delta, t+\delta) = \min_u \left[ \int_{t+\delta}^{t_f} L(x, u, t) dt \right]$$

For a very small  $\delta$

$$f(x, t) = \min_u [L(x, u, t)\delta + f(x+\dot{x}\delta, t+\delta)] + O(\delta^2) \quad \dots\dots(3.31)$$

and the familiar form of the recurrence equation has emerged. Expressing the optimal criterion function as a Taylor Series expansion about the nominal state  $(x, t)$  and assuming the second partial derivatives exist and are bounded in the interval  $(t, t+\delta)$ , provides

$$f(x+\dot{x}\delta, t+\delta) = f(x, t) + \frac{\partial f}{\partial t}\delta + \frac{\partial f}{\partial x}\dot{x}\delta + o(\delta^2) \quad \dots\dots\dots(3.32)$$

and for an infinitesimal  $\delta$  the higher powers of  $\delta$  may be ignored. Thus

$$f(x, t) = \min_u [L(x, u, t)\delta + f(x, t) + \frac{\partial f}{\partial t}\delta + \frac{\partial f}{\partial x}\dot{x}\delta] \quad \dots\dots\dots(3.33)$$

but  $f(x, t)$  is not a function of the control vector  $u$  and may be removed from the minimisation. By dividing throughout by  $\delta$  before allowing it to tend toward zero, results in

$$0 = \min_u [L(x, u, t) + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\dot{x}] \quad \dots\dots\dots(3.34)$$

and the process model

$\dot{x} = \psi(x, u, t)$  provides the functional relationship between state and control vectors to give

$$-\frac{\partial f}{\partial t} = \min_u [L(x, u, t) + \frac{\partial f}{\partial x}\psi(x, u, t)] \quad \dots\dots\dots(3.35)$$

which is the Hamilton-Jacobi equation of classical mathematical literature. For the chosen criterion the boundary conditions are at  $t=t_f$  where

$f(x, t_f) = 0$ , but for the more general criterion of equation 2.22 the terminal condition is

$$f(x, t_f) = F[x(t_f)]$$

Unfortunately the analytical solution of the Hamilton-Jacobi equation is limited to a few special cases where the form of the state equations and the criterion function result in a particular form of optimal criterion.

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### 3.5 DYNAMIC PROGRAMMING AND THE CALCULUS OF VARIATIONS.

Partial differentiation of the right hand side of equation 3.35 with respect to  $u$  and equating to zero defines the minimising control from

$$\frac{\partial L}{\partial u} + \frac{\partial f}{\partial x} \frac{\partial \psi}{\partial u} = 0 \quad \text{.....(3.36)}$$

and substitution back into equation 3.35 completes the non-linear partial differential equation

$$-\frac{\partial f(x,t)}{\partial t} = L(x,u^0,t) + \frac{\partial f}{\partial x} \psi(x,u^0,t) \quad \text{.....(3.37)}$$

which provides the sufficient conditions for optimality (McCausland, 1969, Bellman, 1971).

Furthermore, division of equation 3.36 by the partial differential  $\partial \psi / \partial u$  gives

$$\frac{\partial L}{\partial \dot{x}} + \frac{\partial f}{\partial x} = 0 \quad \text{.....(3.38)}$$

and subsequent differentiation with respect to time

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \frac{\partial^2 f}{\partial x \partial t} + \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} = 0 \quad \text{.....(3.39)}$$

Partial differentiation of equation 3.37 with respect to  $x$  provides

$$\frac{\partial L}{\partial x} + \frac{\partial L}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial x} + \frac{\partial^2 f}{\partial x \partial t} + \frac{\partial^2 f}{\partial x^2} \dot{x} + \frac{\partial f}{\partial x} \frac{\partial \dot{x}}{\partial x} = 0 \quad \text{.....(3.40)}$$

and by combining equations 3.38 and 3.40, and subtracting from equation 3.39 produces

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \text{.....(3.41)}$$

which is the Euler-Lagrange equation of the classical Calculus of Variations.

The relationship between the Hamilton-Jacobi equation and variational mathematics may be further developed by defining an augmented criterion where the constraining state equations are adjoined to the criterion functional, as per equation 2.24, to give

$$J(x, t) = [F(x) + \lambda^T x]_{t_f} + [\lambda^T x]_t + \int_t^{t_f} (L + \dot{\lambda}^T x) dt \quad \dots\dots\dots(3.42)$$

from which

$$\begin{aligned} \left. \frac{\partial J}{\partial t} \right|_x &= [\dot{\lambda}^T x - L - \dot{\lambda}^T x]_t \\ &= -H(x, \lambda, u, t) \quad \dots\dots\dots(3.43) \end{aligned}$$

and

$$\left. \frac{\partial J}{\partial x} \right|_t = \lambda(t) \quad \dots\dots\dots(3.44)$$

The optimal trajectory, denoted by the superscript "o", results in the optimal criterion

$$J^o = J(x^o, t) \quad \text{and from equations 3.43 and 3.44}$$

$$\left. \frac{\partial J^o}{\partial t} \right| = -H(x^o, \lambda^o, u^o) \quad \dots\dots\dots(3.45)$$

$$\left. \frac{\partial J^o}{\partial x} \right| = \lambda^o(t) \quad \dots\dots\dots(3.46)$$

Substitution into the Hamilton-Jacobi equation gives

$$\min_u [L(x^o, u, t) + \lambda^{oT} \varphi(x^o, u, t)] = H(x^o, \lambda^o, u^o)$$

$$\text{or} \quad \min_u [H(x^o, \lambda^o, u)] = H(x^o, \lambda^o, u^o) \quad \dots\dots\dots(3.47)$$

which implies that on an optimal trajectory the optimal value of  $u(t)$  is that which globally minimises the Hamiltonian, which is a restatement of Pontryagin's Minimum Principle.

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### 3.6 ANALYTICAL SOLUTION OF THE HAMILTON-JACOBI EQUATION

As mentioned earlier it is unlikely that an analytical solution exists for the Hamilton-Jacobi equation, but two simplifying assumptions result in a set of equations which are possibly tractable.

(1) If the process can be regarded as of infinite duration, which is generally the case for regulatory control, the optimal criterion functional will be independent of time, i.e.

$$\partial f / \partial t = 0 \quad \text{.....(3.48)}$$

and (2) For a linear process with quadratic criterion, foreshadowing a linear feedback relationship, the optimal criterion functional may be assumed to be of quadratic form, i.e.

$$f(x,t) = K_0(t) + K_1(t)x + x^T K_2(t)x \quad \text{.....(3.49)}$$

Furthermore, where the final state is free and the desired steady-state is

$x^T = (0 \ 0 \ \dots 0)$  the form may be simplified without loss of generality to

$$f(x,t) = x^T K(t)x \quad \text{.....(3.50)}$$

where  $K(t)$  is required to be symmetric and positive-definite to ensure convexity (McCausland, 1969).

For the continuous system expressed in standard linear form

$$\dot{x} = Ax + Bu \quad \text{.....(3.51)}$$

a quadratic criterion, appropriate to regulatory control, is

$$J = \int_{t_0}^{t_f} (x^T Q x + u^T R u) dt \quad \text{.....(3.52)}$$

where  $Q$  is positive semi-definite and  $R$  is positive definite. Using the proposed quadratic form for the optimal criterion functional

$$\begin{aligned} f(x,t) &= x^T K(t)x \\ &= \min_u \int_t^{t_f} L(x,u,t) dt \end{aligned} \quad \text{.....(3.53)}$$

substitution into the Hamilton-Jacobi equation

$$\min_u \left[ L(x, u, t) + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \dot{x} \right] = 0 \quad \dots\dots\dots(3.54)$$

provides the equation

$$\dot{K} = -Q - A^T K - KA + KBR^{-1}B^T K \quad \dots\dots\dots(3.55)$$

which is the continuous Matrix Riccati Equation.

The control law is found from the minimisation of equation 3.54 to be

$$u^o = -R^{-1}B^TK(t)x \quad \dots\dots\dots(3.56)$$

and the predicted linear feed-back form has emerged.

The Riccati equation may be simplified if the first assumption of an infinite time horizon (and therefore a steady-state solution) is valid, since

$$\frac{\partial f(x,t)}{\partial t} = 0 \Rightarrow \dot{K} = 0$$

Books and papers abound with techniques and algorithms for solving Riccati-type equations. In this work a Newton-Raphson technique was investigated for steady-state solutions and direct numerical integration as well as the method of Kalman and Englar were used for dynamic solutions.

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## 3.7 NEWTON-RAPHSON SOLUTION OF THE STEADY-STATE RICCATI EQUATION.

For the steady-state solution  $\dot{K} = 0$ , and the solution matrix  $\bar{K}$  must therefore satisfy the equation

$$F(\bar{K}) = 0 = -Q - A^T \bar{K} - K A + K B R^{-1} B^T \bar{K} \quad \text{.....(3.57)}$$

An iterative procedure is derived to converge on a non-negative definite symmetric matrix  $K$  which satisfies this equation. Supposing an estimate at the  $k$ 'th iteration is  $K_k$  which differs from the true solution by  $\tilde{K}$ , such that  $\bar{K} = K_k + \tilde{K}$ , then if  $\tilde{K}$  is small enough to ignore the quadratic terms

$$F(\bar{K}) = -Q - A^T (K_k + \tilde{K}) - (K_k + \tilde{K}) A + K_k T K_k + K_k T \tilde{K} + \tilde{K} T K_k \quad \text{.....(3.58)}$$

where  $T = B R^{-1} B^T$ .  $\tilde{K}$  may be estimated by setting the right hand side of equation 3.58 to zero. If this estimate of  $\tilde{K}$  is  $\tilde{K}_k$  then the estimate of  $\bar{K}$  may be updated by

$$K_{k+1} = K_k + \tilde{K}_k \quad \text{and equation 3.58 may be rearranged to}$$

give

$$0 = -Q - K_k T K_k - A_k^T K_{k+1} - K_{k+1} A_k \quad \text{.....(3.59)}$$

where

$$A_k = A - T K_k$$

Equation 3.59 is now in a simple linear form and solution is more straightforward. McClamroch (1969) has shown that

$\lim_{k \rightarrow \infty} K_k = \bar{K}$  provided that  $K_0$  is chosen such that the composite matrix

$$A_0 = A - T K_0 \quad \text{is asymptotically stable. For } A \text{ not}$$

asymptotically stable, a condition deliberately chosen in this work, the choice of  $K_0$  may present considerable difficulties and the iterations may either fail to converge or converge to a non-negative definite solution. Kleinman (1970) gives a method for selecting  $K_0$  when  $A$  is not asymptotically stable but in this work the Newton-Raphson method was found to be slow and the uncertain convergence was not acceptable for an in-line control system synthesis.

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### 3.8 DIRECT EULER INTEGRATION OF THE MATRIX RICCATI EQUATION.

Equation 3.55 may be considered as a set of  $n^2$  non-linear differential equations which can be solved simultaneously in reverse time from the terminal conditions  $K(t_f)$ . Euler integration gives

$$K(t+\delta t) \doteq K(t) + \dot{K}(t)\delta t \quad \text{.....(3.60)}$$

which is very simple to implement on a computer. For stability and accuracy, however, the step size  $\delta t$  must be kept small which results in long computing times if the steady-state solution is required. In this work both oscillatory behaviour and irregular convergence were encountered, possibly due to the unstable nature of the open-loop processes, and the technique of Kalman and Enqlar was preferred.

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### 3.9 THE KALMAN-ENGLAR TECHNIQUE.

The method due to Kalman and Enqlar(1966) for the dynamic solution of the Riccati equation uses the co-state variable approach to the optimal control law, i.e.

$$u(t) = -R^{-1}B^T\lambda(t) \quad \text{.....(3.61)}$$

$$\text{for } t_0 \leq t \leq t_f$$

and the co-state equations

$$\dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) \quad \text{.....(3.62)}$$

have boundary conditions only at  $t = t_f$ , where

$$\lambda(t_f) = K(t_f)x(t_f) \quad \text{.....(3.63)}$$

leading to the well-known two point boundary value problem of the Calculus of Variations. The two sets of differential equations may be written in the homogeneous form

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} &= \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \\ &= Z \begin{bmatrix} x \\ \lambda \end{bmatrix} \quad \text{.....(3.64)} \end{aligned}$$

which has the transition matrix

$$\Theta(\delta t) = \exp(-Z\delta t)$$

From equation 3.63 the matrix  $K(t)$  may now be found from

$$K(t+\delta t) = [\Phi_{21}(\delta t) + \Phi_{22}(\delta t)K(t)][\Phi_{11}(\delta t) + \Phi_{12}(\delta t)K(t)]^{-1} \quad \dots\dots(3.65)$$

where the  $\Phi_{ij}$  are obtained by partitioning the matrix  $\Theta$ . For a time invariant system the matrices  $\Phi$  need to be calculated only once and the solution is relatively straightforward. Convergence is generally rapid except for large values of  $\delta t$  where terms in the matrix series evaluation of  $\Theta(\delta t)$  may become very large leading to numerical instabilities. If  $\delta t$  is very small, however, the convergence of equation 3.65 to the steady-state solution may be very slow. Small values of  $\delta t$  are recommended for very "stiff" systems (Vaughan, 1969) but no difficulties were encountered with non-asymptotically stable systems and convergence was always achieved for terminal condition matrices  $K(t_f) \geq 0$ .

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## CHAPTER IV

## FEEDFORWARD CONTROL TECHNIQUES

The advantages of feedforward control are discussed and techniques for the design of multivariable feedforward control systems are developed for continuous and discrete controllers.

	Page
4.1 Introduction to Feedforward Compensation	50
4.2 Continuous Feedforward Control	51
4.3 Feedforward Control for a Discrete Process	54
4.4 Feedforward Control by Dynamic Programming Algorithm	55
4.5 Non-linear Feedforward Compensation	57

## CHAPTER IV

### FEEDFORWARD CONTROL TECHNIQUES

#### 4.1 INTRODUCTION TO FEEDFORWARD COMPENSATION.

Where a disturbance can be detected and measured before it enters the system it is possible to take control action to reduce the effect of the disturbance on the stability of the process or the quality of the product. Because feedforward control can take effect before the whole system is upset by the disturbance it has a major advantage over feedback control where the disturbance has to work its way through to the system outputs before corrective action can be taken.

Feedforward control does require that the process inputs be measured but for integrated plant the input variables for one plant item will be the output variables from another and would probably have been measured for feedback control purposes anyway.

Because feedforward control is not a self-regulating form of compensation it is almost always used in conjunction with a feedback controller in order that inaccuracies or non-linearities in the process model do not result in permanent process variable offsets. Feedforward control may be of either static or dynamic type and may be implemented in single-loop, multi-loop or multivariable form. With static feed forward control the outputs are directly proportional to the inputs and the controller is designed to eliminate or minimise the steady-state offsets of the process variables in the face of sustained load variable disturbances.

The same steady-state objectives are implemented with dynamic feedforward control but dynamic elements complementary to the disturbance and process dynamics are also included, usually involving the ubiquitous "lead-lag" element.

With feedforward control, as with feedback, the control loop config-

uration is a major problem where single- or multi-loop strategies are used on a multivariable process. The design techniques are reasonably straightforward once the configuration has been chosen or if the multivariable approach is used.

The theory of feed forward control is presented by Shinskey (1967), Smith (1972) and Luyben (1973), and design examples are detailed by Shinskey (1963), McMullen and Shinskey (1964), Nisenfeld and Hoyle (1970), Newell (1971) and Kim (1975). Bollinger and Lamb (1962) and Bruun (1975) present the design procedure for the multivariable case in laplace domain and demonstrate the development of the lead-lag transfer functions. These are also transformed into the discrete time domain for implementation by digital process controllers. Bruun also applies feedforward control to a five-state double-effect evaporator and demonstrates excellent steady-state compensation for step changes in feed flow rate and concentration.

Newell (1971) develops both continuous and discrete multivariable feedforward controllers and demonstrates their effectiveness on a double-effect evaporator. Johnson (1970) and Sobral and Stefanek (1970) in studies of linear regulators subject to disturbances demonstrate that the effect of disturbances may be eliminated by a transformation of the control problem such that the compensation is generated by the process states.

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#### 4.2 CONTINUOUS FEEDFORWARD CONTROL.

For a process represented in continuous linear state variable form

$$\dot{x} = Ax + Bu + Cd \quad \text{.....(4.1)}$$

the linear control law comprises both feedback and feedforward components

$$u = u_{FB} + u_{FF} \quad \text{.....(4.2)}$$

The feedback control law, designed by the techniques presented earlier, will be of the form

$$u_{FB} = K_{FB} x \quad \text{and the closed-loop representation of the}$$

process, with no feedforward control, is

$$\dot{x} = (A + BK_{FB})x + Cd \quad \text{.....(4.3)}$$

so that the steady-state offsets are given by

$$x_0 = -(A+BK_{FB})^{-1} Cd \quad \text{.....(4.4)}$$

Proposing an admissible control  $u_{FF}$  such that equilibrium may be achieved at  $\dot{x} = x = 0$ , then from equation 4.1

$$0 = Bu_{FF} + Cd \quad \text{.....(4.5)}$$

and if the range of  $C$  is contained within the range of  $B$ , such that

$$C = -BK_{FF}, \text{ then the control law is } u_{FF} = K_{FF}d.$$

For  $B$  of rank  $m(<n)$ , if  $K_{FF}$  exists it is uniquely defined by

$$K_{FF} = -(B^T B)^{-1} B^T C \quad \text{.....(4.6)}$$

provided that the system matrices  $[A;B]$  constitute a controllable pair.

It is clear that for  $n$  states and  $m$  controls there are only  $m$  degrees of freedom and it would be possible to achieve zero offsets in only  $q$  states where  $q \leq m$ . If  $q < m$  it is possible to use the remaining degrees of freedom to minimise a functional of the remaining states. For a given measurable disturbance  $d$  the desired steady-state for a sub-set of states  $x_1$  is taken to be  $x_{1d}$ , such that for  $\dot{x} = 0, x_1 = x_{1d}$ . The process model may be partitioned to show

$$0 = [A_1 \ A_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [B_1 \ B_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Cd \quad \text{.....(4.7)}$$

and rearranged to give

$$[B_1 \ A_2] \begin{bmatrix} u_1 \\ x_2 \end{bmatrix} = -B_2 u_2 - A_1 x_{1d} - Cd \quad \text{.....(4.8)}$$

from which a control law of the form

$$u_1 = K_1 u_2 + K_2 x_{1d} + K_3 d \quad \text{.....(4.9)}$$

may be obtained. This is similar to the development of Anderson(1969) except that he used the closed-loop form for the matrix  $A$ , and solved for  $q = m$  only. Where the dimension of  $u_2$  is non-zero it is possible to substitute equation 4.9 back into 4.1 to get

$$\begin{aligned} \dot{x} &= [A_1 \ A_2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [B_1 K_1 + B_2] u_2 \\ &\quad + [B_1 K_3 + C] d + B_1 K_2 x_{1d} \quad \text{.....(4.10)} \end{aligned}$$

and the control law for  $u_2$  may be found by dynamic programming techniques to minimise the criterion functional  $J = J(x_2, u_2)$

Kim (1975) has shown that for an unmeasurable but constant disturbance vector a P-I controller is optimal, following the work of Johnson (1970) and Sobral and Stefanek (1970). Employing equation 4.5 for conformable matrices B,C gives

$$\dot{x} = Ax + B(u_{FB} + K_{FF}d) + Cd \quad \dots\dots(4.11)$$

If the state is augmented with a vector  $y$ , where

$$y = u_{FB} + K_{FF}d$$

then  $y(0) = u(0) + K_{FF}d$  and because of the constant disturbance vector  $d$

$$\dot{y} = \dot{u} = v$$

The augmented equation is therefore

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} v \quad \text{and } v \text{ may be chosen to}$$

minimise a quadratic functional resulting in the familiar linear feedback law

$$v = -\bar{K} \begin{bmatrix} x \\ y \end{bmatrix}$$

Therefore

$$u = u(0) + \int_0^t v(\tau) d\tau \quad \dots\dots(4.12)$$

resulting in the linear dynamic process model

$$\dot{x} = Ax + B\left[u(0) + \int_0^t v(\tau) d\tau\right] + Cd \quad \dots\dots(4.13)$$

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### 4.3 FEEDFORWARD CONTROL FOR A DISCRETE PROCESS.

For the discrete-time linear process described by

$$x_{k+1} = \phi x_k + \Delta u_k + \psi d_k \quad \dots\dots\dots(4.14)$$

and a linear control law of the form

$$u_k = K_{FB} x_k + u_{FFk} \quad \dots\dots\dots(4.15)$$

the closed-loop form of equation 4.11 becomes

$$x_{k+1} = (\phi + \Delta K_{FB}) x_k + \Delta u_{FFk} + \psi d_k \quad \dots\dots\dots(4.16)$$

For a constant disturbance vector  $d$  the process steady-state will be

$$x_{k+1} = x_k = x_s \text{ as } k \rightarrow \infty \text{ and}$$

$$x_s = -(\mathbf{I} - \phi - \Delta K_{FB})^{-1} (\Delta u_{FF} + \psi d_s) \quad \dots\dots\dots(4.17)$$

Assuming that  $\Delta^{-1}$  exists the feedforward control vector will be defined by

$$u_{FF} = -\Delta^{-1}(\mathbf{I} - \phi - \Delta K_{FB}) x_s - \Delta^{-1} \psi d_s \quad \dots\dots\dots(4.18)$$

and for a desired steady-state of  $x_s = 0$  the feedforward control vector becomes

$$u_{FFk} = -\Delta^{-1} \psi d_k \quad \dots\dots\dots(4.19)$$

As with the continuous case, zero steady-state offsets may not be achieved for all states unless  $m \geq n$  but a functional of steady-state offsets may be minimised. For a quadratic criterion functional

$$J = x_s^T Q x_s \quad \dots\dots\dots(4.20)$$

where  $Q$  is positive semi-definite, the steady-state offsets  $x_s$  will be given by the closed-loop form of equation 4.17. Differentiation of the criterion, equation 4.20, with respect to the feedforward control vector  $u_{FF}$  and equating the set of partial differential equations to zero,

$$\frac{1}{2} \frac{\partial J}{\partial u_{FF}} = \frac{\partial x_s}{\partial u_{FF}} Q x_s = 0 \quad \dots\dots\dots(4.21)$$

means that the minimising controls may be found from the set of linear equations

$$u_{FF} = -(\Delta^T \Gamma^T Q \Gamma \Delta)^{-1} (\Delta^T \Gamma^T Q \Gamma \psi) d_k \quad \dots\dots\dots(4.22)$$

where  $\Gamma = -(I - \Phi - \Delta K_{FB})^{-1}$

Once again it can be shown that for  $m = n$  and non-singular  $\Delta$  the minimising control law is given by

$$u_{FF_k} = -\Delta^{-1} \Psi d_k \quad \dots\dots(4.23)$$

and compensation may be achieved with zero offsets in all states.

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#### 4.4 FEEDFORWARD CONTROL BY DYNAMIC PROGRAMMING ALGORITHM.

The Dynamic Programming recurrence equations for dynamic or steady-state feedback gain matrices may be augmented to include a feedforward control algorithm. Employing the approach of section 3.3 the quadratic criterion

$$J = x_N^T S x_N + \sum_{k=0}^N (x_k^T Q x_k + u_k^T R u_k) \quad \dots\dots(4.24)$$

is to be minimised with respect to a control variable comprising both feedforward and feedback components, i.e.

$$u_k = K_{FB} x_k + K_{FF} d_k \quad \dots\dots(4.25)$$

The sequence may be initiated by optimising over the  $N$ 'th time interval, controlled by  $u_N$ .

$$\frac{\partial J}{\partial u_N} = \frac{\partial}{\partial u_N} [x_N^T (Q + S) x_N + u_N^T R u_N] \quad \dots\dots(4.26)$$

By setting the partial differential equation to zero

$$\frac{\partial x_N}{\partial u_N} (Q + S) x_N + R u_N = 0 \quad \dots\dots(4.27)$$

provides

$$\Delta^T (Q + S) \Phi x_N + [\Delta^T (Q + S) \Delta + R] u_N + \Delta^T (Q + S) \Psi d_N = 0 \quad \dots\dots(4.28)$$

and the control law for the  $N^{\text{th}}$  stage is

$$u_N = K_{FB_N} x_N + K_{FF_N} d_N \quad \text{where}$$

$$K_{FB_N} = -[\Delta^T(Q + S)\Delta + R]^{-1} \Delta^T(Q + S)\Phi \quad \dots\dots\dots(4.29)$$

and

$$K_{FF_N} = -[\Delta^T(Q + S)\Delta + R]^{-1} \Delta^T(Q + S)\Psi \quad \dots\dots\dots(4.30)$$

Considering the  $N-1^{\text{th}}$  and  $N-2^{\text{th}}$  stages, etc, the recursive relationship is found to give

$$K_{FB_k} = -[\Delta^T P_k \Delta + R]^{-1} \Delta^T P_k \Phi \quad \dots\dots\dots(4.31)$$

and

$$K_{FF_k} = -[\Delta^T P_k \Delta + R]^{-1} [\Delta^T P_k \Psi + \Delta^T M_k] \quad \dots\dots\dots(4.32)$$

where

$$P_k = L_k^T P_{k+1} L_k + K_{FB_k}^T R K_{FB_k} + Q \quad \dots\dots\dots(4.33)$$

$$M_k = L_k^T M_{k+1} + L_k P_{k+1} (\Delta K_{FF_{k+1}} + \Psi) + K_{FB_{k+1}}^T R K_{FF_{k+1}} \quad \dots\dots\dots(4.34)$$

and  $L_k$  is the closed-loop matrix for the  $k^{\text{th}}$  stage

$$L_k = \Phi + \Delta K_{FB_k}$$

The matrices  $P$  and  $M$  are initialised by

$$P_N = Q + S \quad \dots\dots\dots(4.35)$$

and

$$M_N = 0 \quad \dots\dots\dots(4.36)$$

Equations 4.31 to 4.34 define the iterative sequence which generates the dynamic feedback and feedforward gains, and convergence to the steady-state matrices is usually rapid. It should be noted that whereas the feedback matrix  $K_{FB}$  is the same as that developed for feedback control alone, the feedforward matrix  $K_{FF}$  is dependent on the feedback matrix and may not be used in isolation.

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#### 4.5 NON-LINEAR FEEDFORWARD COMPENSATION.

With access to the computational power of an on-line process control computer non-linear feedforward compensation is feasible if a reasonably accurate model of the process is available. Since a steady-state compensation is required the technique involves setting the state derivatives to zero and solving for the control variables in terms of the desired state variables and the measured disturbances. Since, however, each control variable may appear in several state equations the control computer may be faced with the solution of sets of implicit non-linear simultaneous equations. Iterative techniques would probably be required and the control strategy would no longer be suitable for on-line control.

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## CHAPTER V

### OPTIMAL FEEDBACK CONTROL OF LINEAR SYSTEMS

Stable and unstable linear second-order models are used to demonstrate the performance of Multivariable optimal feedback control algorithms. The discrete interval is shown to have a significant effect on the performance of the system, particularly for the unstable process model. Single-loop controllers did not compare favourably with the Multivariable controllers in optimising the control criterion or in stabilising the process with a bounded control variable.

	Page
5.1 Continuous System Models	60
5.2 Discrete Time Systems	64
5.3 The Effects of Discrete Time Interval on System Performance	69
5.4 Optimality of the Feedback Laws	71
5.5 The Effects of Control Variable Bounds	71
5.6 Sensitivity of the System to Perturbations of the Feedback Matrix	76
5.7 Comparison of Multivariable and Single-Loop Controllers	77
5.8 Limitation of Control Action by the Control Penalty Matrix	80

## CHAPTER V

## OPTIMAL FEEDBACK CONTROL OF LINEAR SYSTEMS

## 5.1 CONTINUOUS SYSTEM MODELS.

In order to assess the performance of the feedback control algorithms an initial range of experiments was designed using simple linear second-order systems. It may be argued that linear systems of such low order bear little relationship to any modern chemical plant, however it is very difficult to keep track of all the parameters and variables of a more realistic model, whereas the simple cause and effect relationships are readily apparent on a small model.

Two models were investigated; one stable:

$$\dot{x} = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & -1.0 \end{bmatrix} x + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} u \quad \dots\dots(5.1)$$

referred to as System "A", and the other, System "B", unstable:

$$\dot{x} = \begin{bmatrix} 0.0 & 1.0 \\ -1.0 & 1.0 \end{bmatrix} x + \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} u \quad \dots\dots(5.2)$$

Each model has only one control variable and the linear feedback law is of the form

$$u = \begin{bmatrix} k_{11} & k_{12} \end{bmatrix} x \quad \dots\dots(5.3)$$

The open-loop eigenvalues of systems "A" and "B" are  $(-0.50 \pm 0.866i)$  and  $(+0.50 \pm 0.866i)$  respectively, indicating an oscillatory response. The process control criterion is the standard quadratic form

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad \dots\dots(5.4)$$

and the minimisation of the criterion leads to the continuous Riccati equation which can be solved analytically, using the Dynamic Programming approach, or iteratively by Newton-Raphson, Euler or the algorithm of Kalman & Englar. The analytical approach is tedious, even for systems of low order, and often an analytical solution cannot be found. The algorithm of

Kalman and Englar was the preferred method, being stable over very wide ranges of all the tested variables, although very short integration steps resulted in an excessive number of iterations to reach a steady-state solution. Table 5.1 lists the number of iterations to calculate both the augmented matrix exponential and the solution to the Matrix Ricatti equation for a range of time intervals. The iterations for the matrix exponential consist of only one matrix multiplication and addition of a  $2n \times 2n$  matrix, whereas each iteration to solve the Ricatti equation involves an inversion, three multiplications, and two additions of an  $n \times n$  matrix. An interval of  $\Delta t = 1.0$  was chosen, although automatic interval halving was occasionally invoked by the exponentiation routine when large coefficients in the augmented matrix would have lead to round-off errors.

TABLE 5.1

Comparison of the number of iterations required to reach steady-state solutions for the augmented matrix  $\exp(Z\Delta t)$  and the matrix Riccati equation for a range of time intervals.

Time interval $\Delta t$	Iterations for the exponential $\exp(Z\Delta t)$	Iterations for the Riccati equation
0.01	8	>200
0.03	9	>200
0.1	11	61
0.3	16	20
1.0	22	5
3.0	34	4

For a single control variable the control penalty matrix  $R$  is a scalar coefficient which must be positive.  $R$  was varied from 0.001 to 100.0 and the effect on the feedback matrix for system "A" is shown in figure 5.1. An increasing penalty on the extent of the control action results in decreasing feedback control coefficients. An investigation of the variation of the closed-loop eigenvalues, figure 5.2, indicates that a



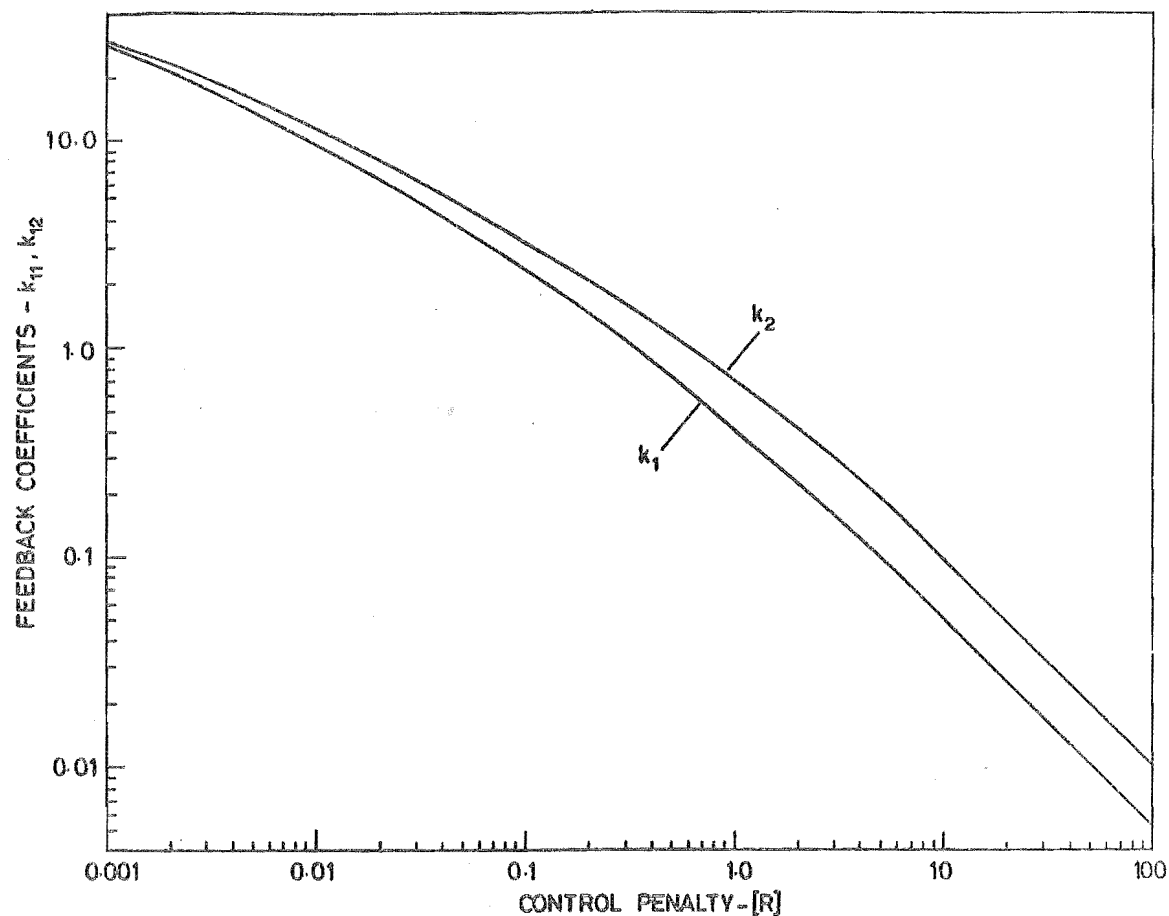


FIGURE 5.1 THE EFFECT OF THE CONTROL PENALTY MATRIX  $R$  ON THE MAGNITUDE OF THE FEEDBACK COEFFICIENTS FOR SYSTEM "A".

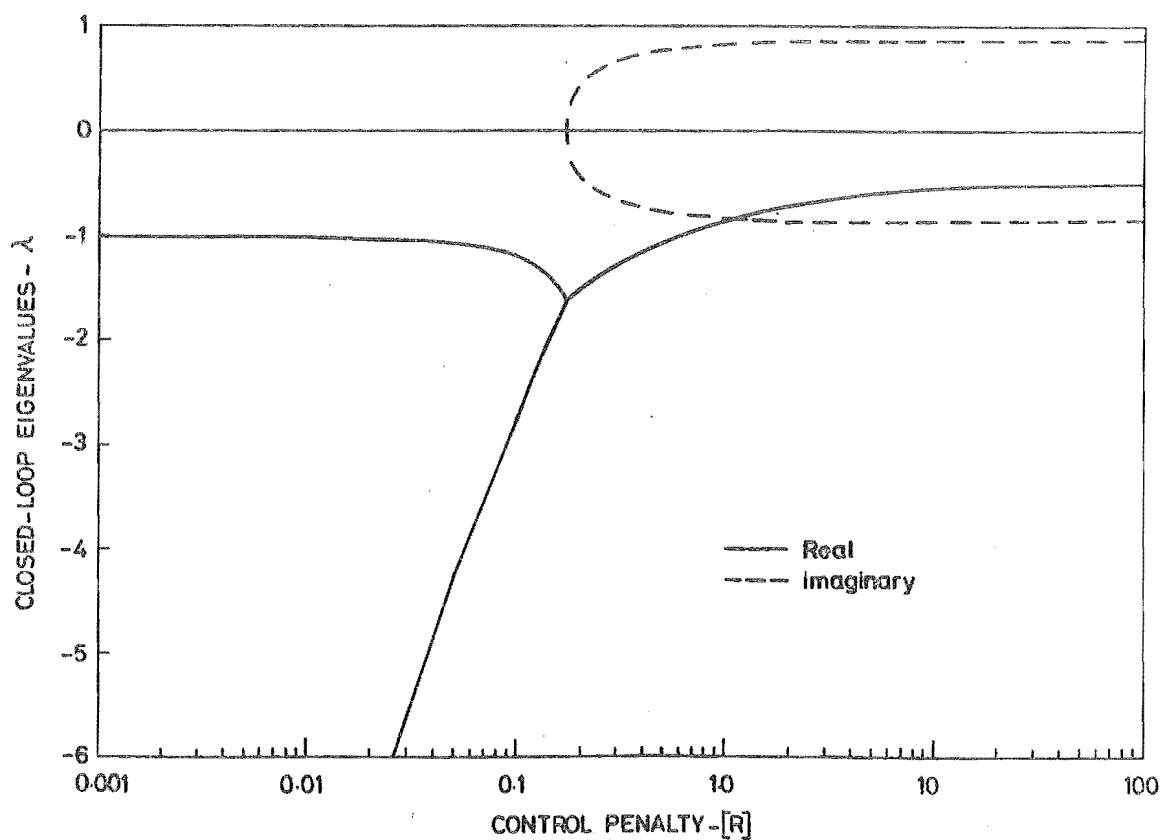


FIGURE 5.2 THE EFFECT OF THE CONTROL PENALTY MATRIX  $R$  ON THE CLOSED-LOOP EIGENVALUES OF SYSTEM "A".

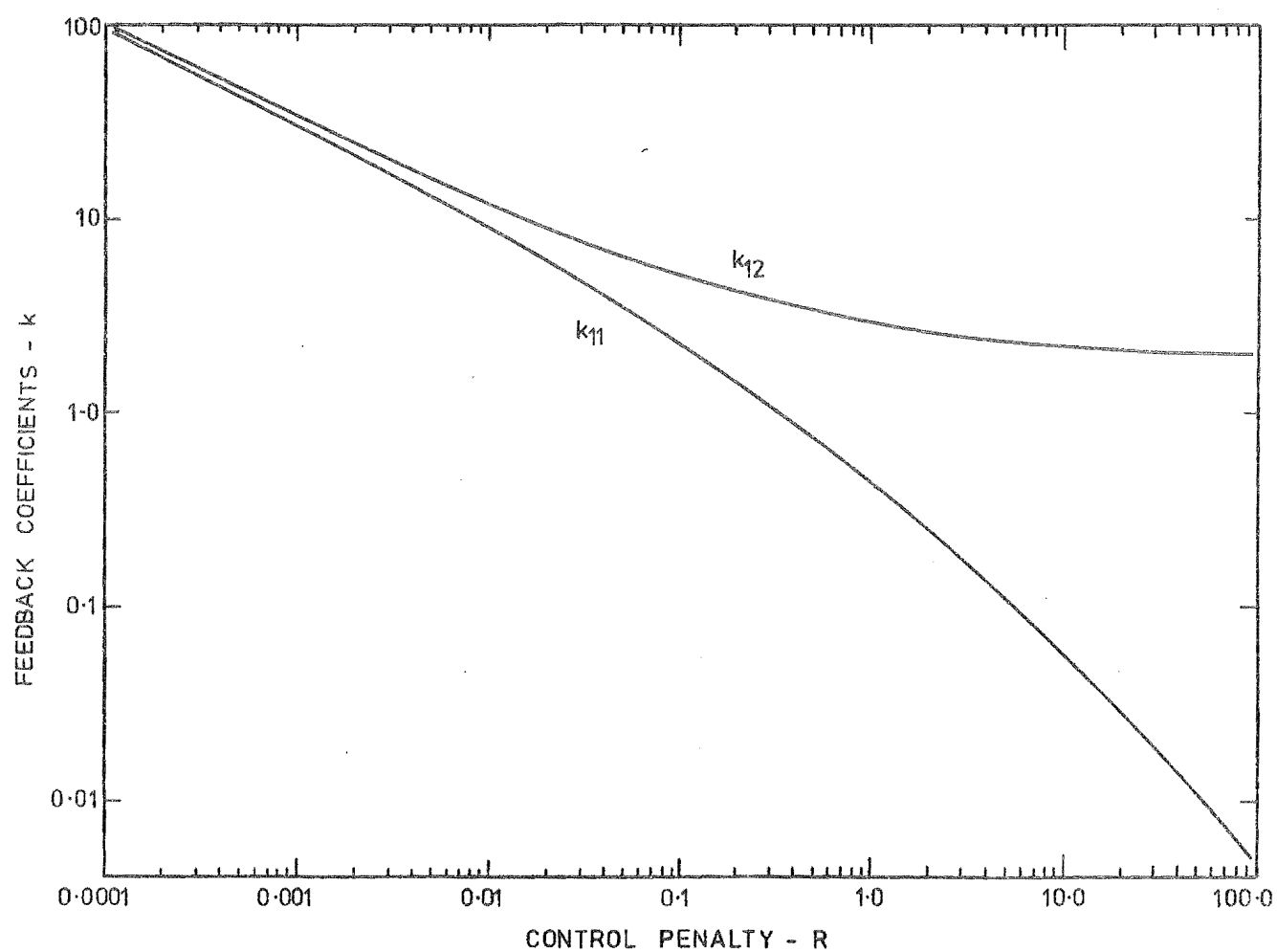


FIGURE 5.3 THE EFFECT OF THE CONTROL PENALTY MATRIX  $R$  ON THE MAGNITUDE OF THE FEEDBACK COEFFICIENTS FOR SYSTEM "B".

very small penalty on the control action, with consequently large feedback coefficients, results in two real and negative eigenvalues indicating an overdamped, non-oscillatory response.

The response becomes oscillatory as the control penalty  $R$  increases past 0.17 and as the penalty becomes very large the closed-loop eigenvalues approach asymptotically the eigenvalues of the open-loop system as expected.

The situation is different for the unstable system "B". No matter what penalty is imposed upon the control action the overall magnitude of the feedback coefficient must be such that the eigenvalues of the closed-loop system are moved to the negative side of the imaginary axis to ensure stability. Figure 5.3 shows feedback coefficient  $k_{11}$  decreasing to a negligibly small value as the control penalty increases, but coefficient  $k_{12}$  approaches the value 2.0 asymptotically. The "optimal" feedback control for both systems "A" and "B" result in identical closed-loop systems, however the magnitude of the control variables used to control the two systems will be considerably different.

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## 5.2 DISCRETE TIME SYSTEMS.

Process control by digital computer is essentially a "sampled data" system; the only knowledge the controller has of the system are the values of the variables at the sampling instants. In order to apply the optimal control algorithms the discrete model of the process is required which describes the system trajectory at the sampling intervals only. The choice of the discrete time interval,  $T$ , has a very considerable effect on the performance of the system. Once it has been chosen, however, the continuous model is used to create a discrete model by solving for the state of the system at time  $(t+T)$  given the values of the process variables at time  $t$  and assuming piecewise constant inputs in the interval  $(t, t+T)$ . The solution involves the calculation of the matrix exponential,  $\exp(AT)$ , and the number

of terms needed for convergence of the matrix series increases with  $T$ , as shown in table 5.2, and was identical for systems "A" and "B".

TABLE 5.2

The variation in the number of iterations to calculate the matrix exponential and the steady-state feedback matrix for the discrete models of systems "A" and "B".

Discrete time Interval $T$	Iterations for the matrix exponential $\exp(AT)$	Iterations for the feedback matrix $K$
0.01	5	>500
0.01	5	>500
0.03	6	216
0.1	7	66
0.3	9	23
0.3	9	23
1.0	14	7
3.0	22	6
10.0	57	4

The steady-state solution of the feedback matrix was calculated using the dynamic programming algorithm and the number of iterations required to reach steady-state decreased with increasing discrete time interval. The number of iterations was similar for systems "A" and "B" with small differences at relatively large values of  $T$ , of which more later.

For this part of the investigation the state and control variable penalty matrices were

$$Q = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad \text{and} \quad R = [0.1]$$

For model "A", the variation of the feedback matrix coefficients with discrete time interval can be seen in figure 5.4. For very small values of

discrete interval the feedback coefficients approach the values of the continuous system, confirming intuitive expectations. As the discrete interval increases the feedback coefficients decrease and oscillate about zero, as expected for an oscillatory process. It is important to note that the decrease in size of the feedback coefficients was not induced by any increase in the size of the control penalty, although at any given value of discrete interval an increase in the control penalty would result in a decrease in the magnitude of the feedback coefficients.

The behaviour of the system may be judged from the eigenvalues of the closed-loop system, figure 5.5, indicating an oscillatory response for  $0.7 < T < 2.2$  and also  $4.6 < T < 6.3$ .

For system "B", apart from an overall greater magnitude, the feedback coefficients initially follow a similar trend of decreasing magnitude and change of sign as for "A", figure 5.6. As the discrete interval increases past 3.0 the coefficient  $k_{12}$  becomes very large, negative and eventually tends toward  $-\infty$  as  $T$  approaches 3.65. As  $T$  increases beyond 3.65  $k_{12}$  reappears again from the direction of  $+\infty$ , crosses the axis at a discrete interval of about 6.0 and again tends toward  $-\infty$  as  $T$  approaches 7.3, establishing a curve similar in shape and periodicity to a plot of the tangent of an angle. Control of the process at these particular "critical" time intervals is obviously very difficult, requiring a very large control effort to bring the system back to steady-state after an upset. Knowledge of the existence of such critical discrete intervals would be crucial to the performance of a control system, and the operation of a discrete controller in the vicinity of such a critical control interval should obviously be avoided.

The closed-loop system is nevertheless still optimal in the vicinity of these critical intervals, although the size of the feedback gain would result in quite unrealistic values for the control variable. As the critical time intervals are approached the system becomes extremely sensitive to perturbations of the model parameters and feedback matrix elements. The closed-loop eigenvalues for "B", as for the continuous model, are indistinguishable from those of "A" up to a time interval of about  $T = 8.0$ , and show no discontinuity about the critical time intervals.

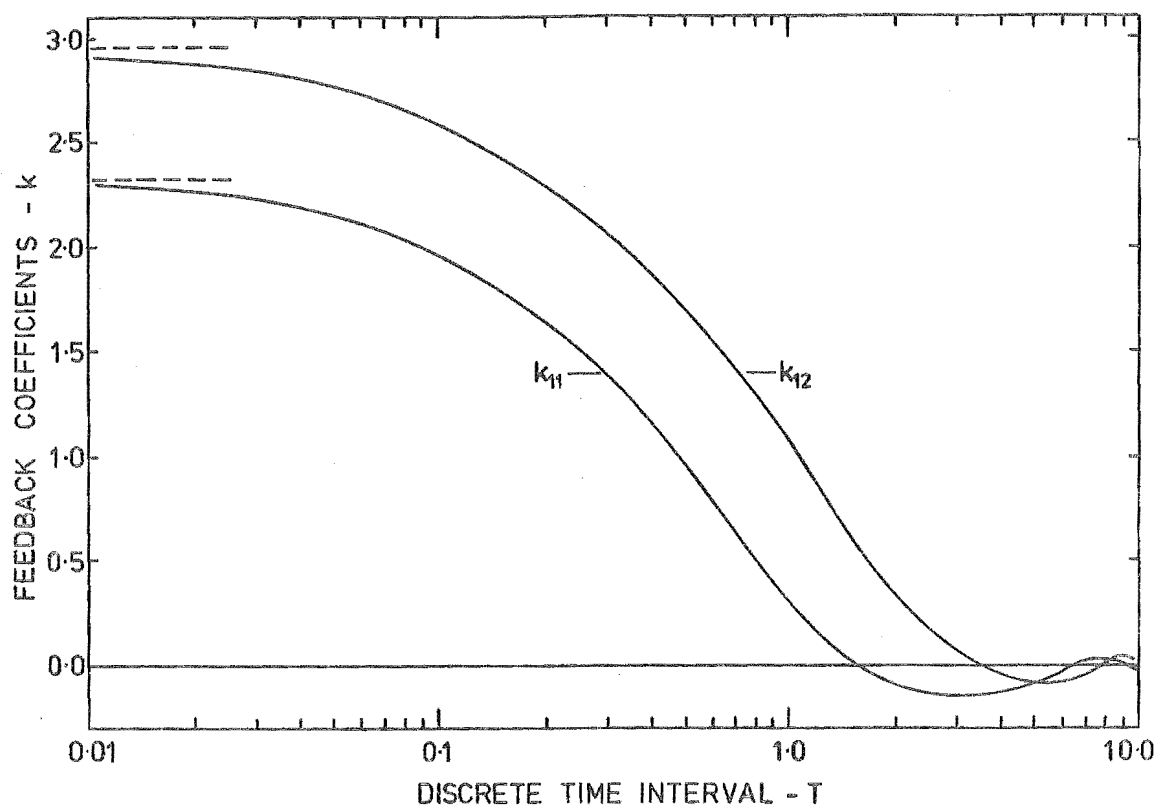


FIGURE 5.4 THE VARIATION OF THE DISCRETE FEEDBACK COEFFICIENTS WITH INCREASING DISCRETE TIME INTERVAL  $T$  FOR SYSTEM "A".

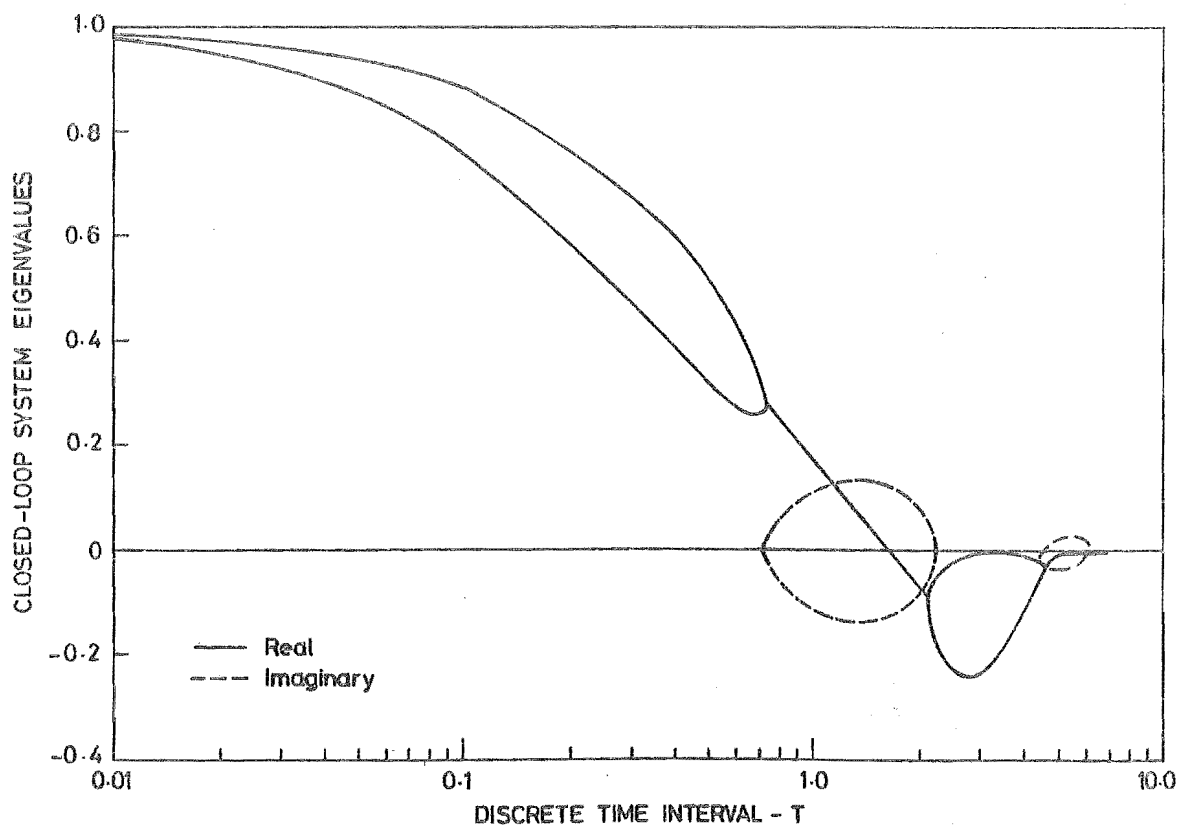


FIGURE 5.5 THE EFFECT OF DISCRETE INTERVAL ON THE CLOSED-LOOP EIGENVALUES OF SYSTEM "A" UNDER OPTIMAL CONTROL.

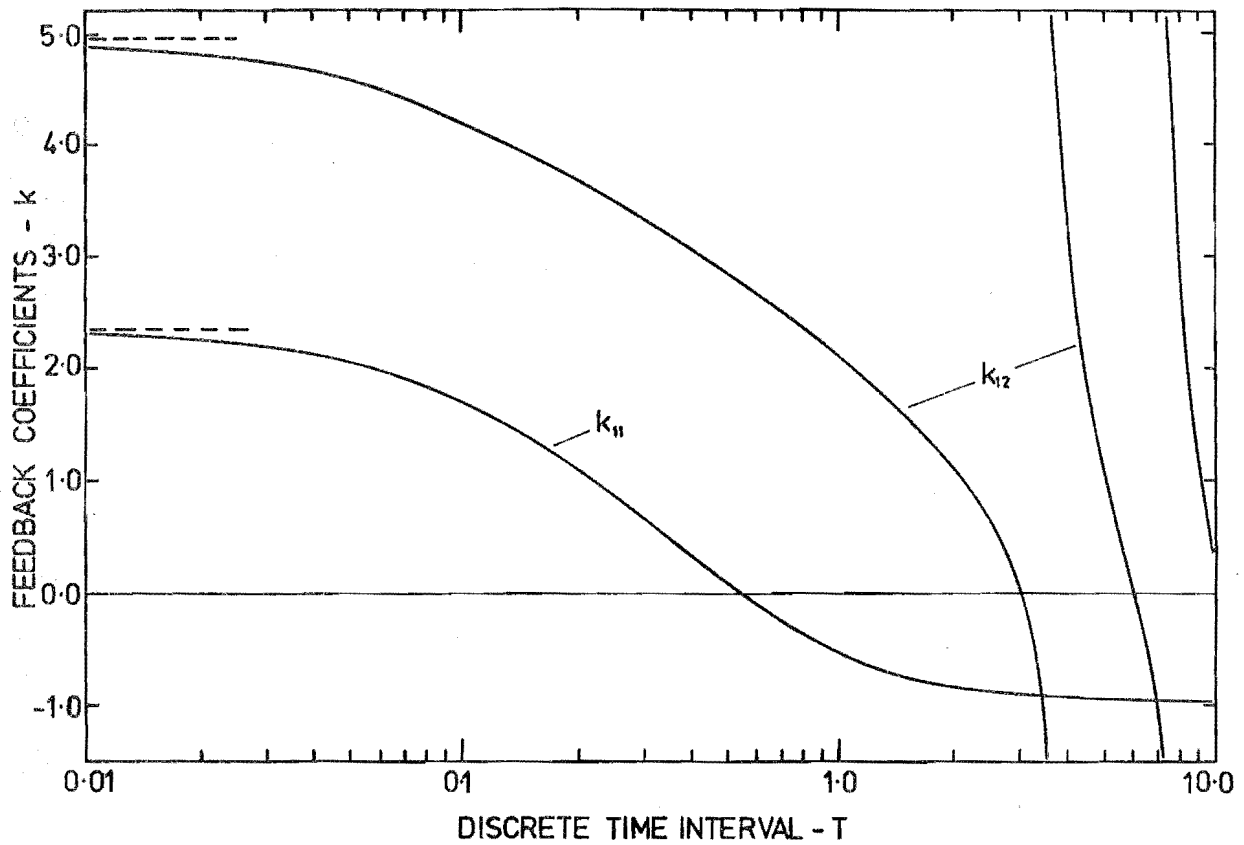


FIGURE 5.6 THE EFFECT OF DISCRETE TIME INTERVAL  $T$  ON THE FEEDBACK COEFFICIENTS FOR SYSTEM "B" SHOWING DISCONTINUITIES AT  $T = 3.65$  AND  $T = 7.3$ .

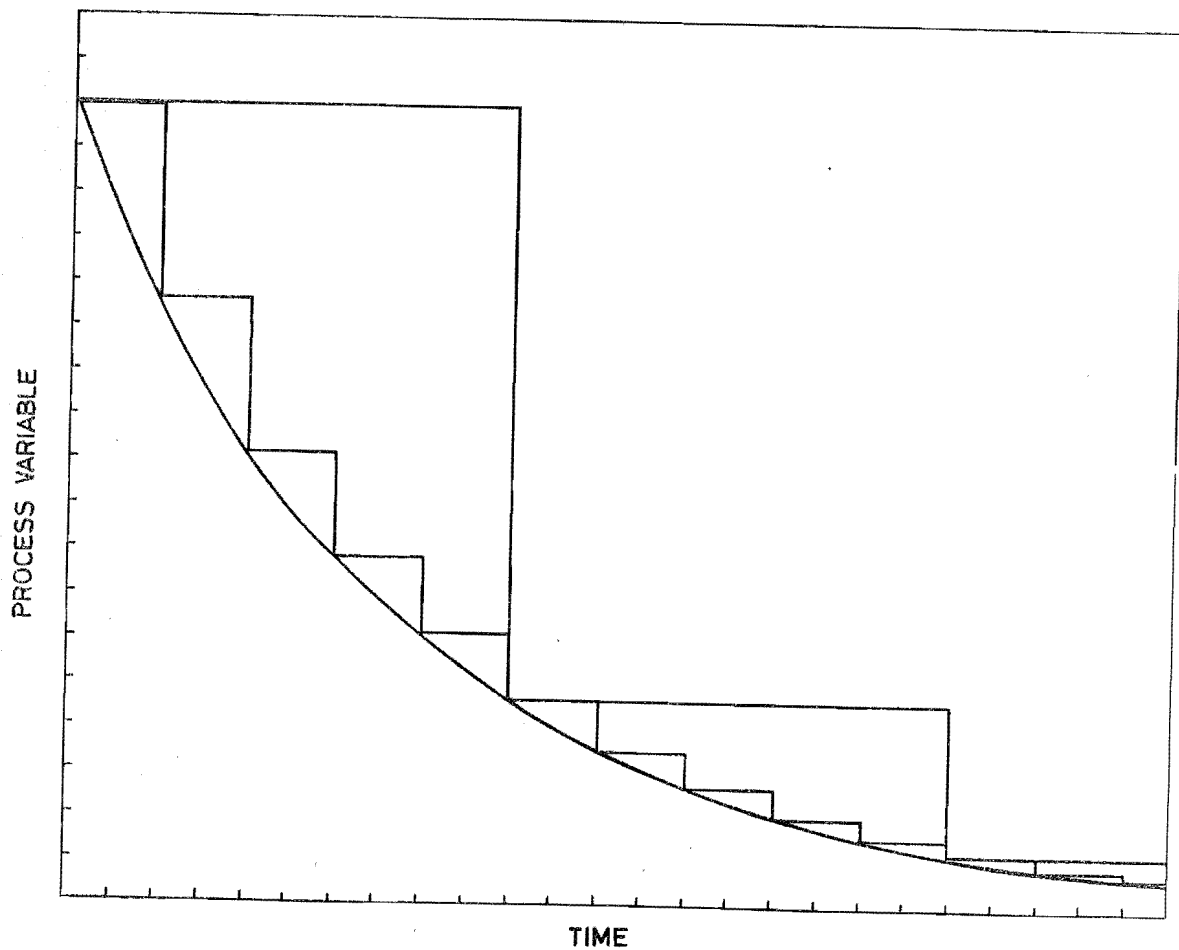


FIGURE 5.7 THE DIFFERENCE IN MEASURED AREA BENEATH THE PROCESS VARIABLE CURVE FOR CONTINUOUS AND DISCRETE PROCESS MODELS.

### 5.3 THE EFFECTS OF DISCRETE TIME INTERVAL ON SYSTEM PERFORMANCE.

For discrete control the control performance criterion becomes

$$J = \sum_{k=1}^{\infty} [x_k^T Q x_k + u_k^T R u_k] \quad \text{.....(5.5)}$$

for control with an infinite horizon, although we may assume a rather shorter time horizon for stable processes in the interests of shorter computing times. Table 5.3 shows the very considerable difference in the performance criterion for the open-loop process "A" with varying discrete time intervals and a total process time of 20.0.

TABLE 5.3

Variation of control criterion with discrete interval for the stable process model "A" with no control action.

Discrete time interval T	Open-loop process Criterion J	Euclidean norm of state vector at time t = 20.0
Continuous	3.50	$1.11 \times 10^{-9}$
0.05	3.55	$1.11 \times 10^{-9}$
0.1	3.60	$1.11 \times 10^{-9}$
0.5	4.04	$1.11 \times 10^{-9}$
1.0	4.64	$1.11 \times 10^{-9}$
5.0	10.18	$1.11 \times 10^{-9}$

There is an apparent degradation in performance which is quite illusory since the process trajectory is identical in each case. Figure 5.7 displays the difference in area between two discrete models with different time intervals and a continuous model for a decreasing exponential curve. Since the process criterion is based on the squared deviations from the desired value the discrepancy between the continuous and discrete model



criteria becomes even more pronounced. For an unstable process the situation is reversed and the process criterion for a discrete system with large discrete interval is significantly smaller. In order to fairly compare the performance of systems with different discrete intervals the process criterion has, throughout this work, been based upon the continuous system. It must be noted, however, that this is not the criterion which was optimised in calculating the optimal feedback matrix since the trajectory between sampling intervals is assumed to be unknown for a discrete system.

Table 5.4 indicates the real and dramatic degradation in the performance of system "B" under closed-loop control with an increasing discrete interval. The process control criterion was based on the continuous system.

TABLE 5.4

Degradation of control quality with increasing discrete interval, using system "B" and a process duration of 20.0.

Discrete time interval T	Optimal closed-loop Criterion J
0.01	2.35
0.1	2.45
0.3	2.5
1.0	4.0
2.0	13.5
3.0	175.0

.....0000000000000000.....

#### 5.4 OPTIMALITY OF THE FEEDBACK LAWS.

The optimal feedback gain matrices were tested for optimality by simulating the processes with the feedback coefficients perturbed by between  $\pm 1.0$  per cent and  $\pm 10.0$  per cent. The calculated feedback matrices were optimal for every system tested, both continuous and discrete.

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#### 5.5 THE EFFECTS OF CONTROL VARIABLE BOUNDS.

The calculation of the optimal feedback matrix using the Quadratic performance criterion does not accommodate bounds on the control variable although it is obvious that unless the control variable is very heavily penalised, or unless the process variable deviations are very small, the control variable is liable to "saturate". If this occurs the linearity of the system will be lost, and the eigenvalues of the closed-loop system are no longer an indication as to the nature of the response. In many published articles on the application of optimal control the penalty matrices for both state and control variables have been manipulated to prevent saturation of the control variable as though it were some sort of sacred barrier. In general this is very far from the truth and control variable saturation, corresponding to a control valve being fully open or closed for example, is no cause for concern. Indeed considerable control action may be lost by arbitrarily preventing the control variable from exercising over its fullest range. The performance of systems "A" and "B" were investigated under conditions of restricted control action.

System "A" shows a relatively gradual increase in process control criterion as the control action is reduced, figure 5.8, with the criterion approaching that for the open-loop response. For the continuous system, and for the discrete system with short time intervals, the feedback control is relatively large and the restriction of the control variable has a more

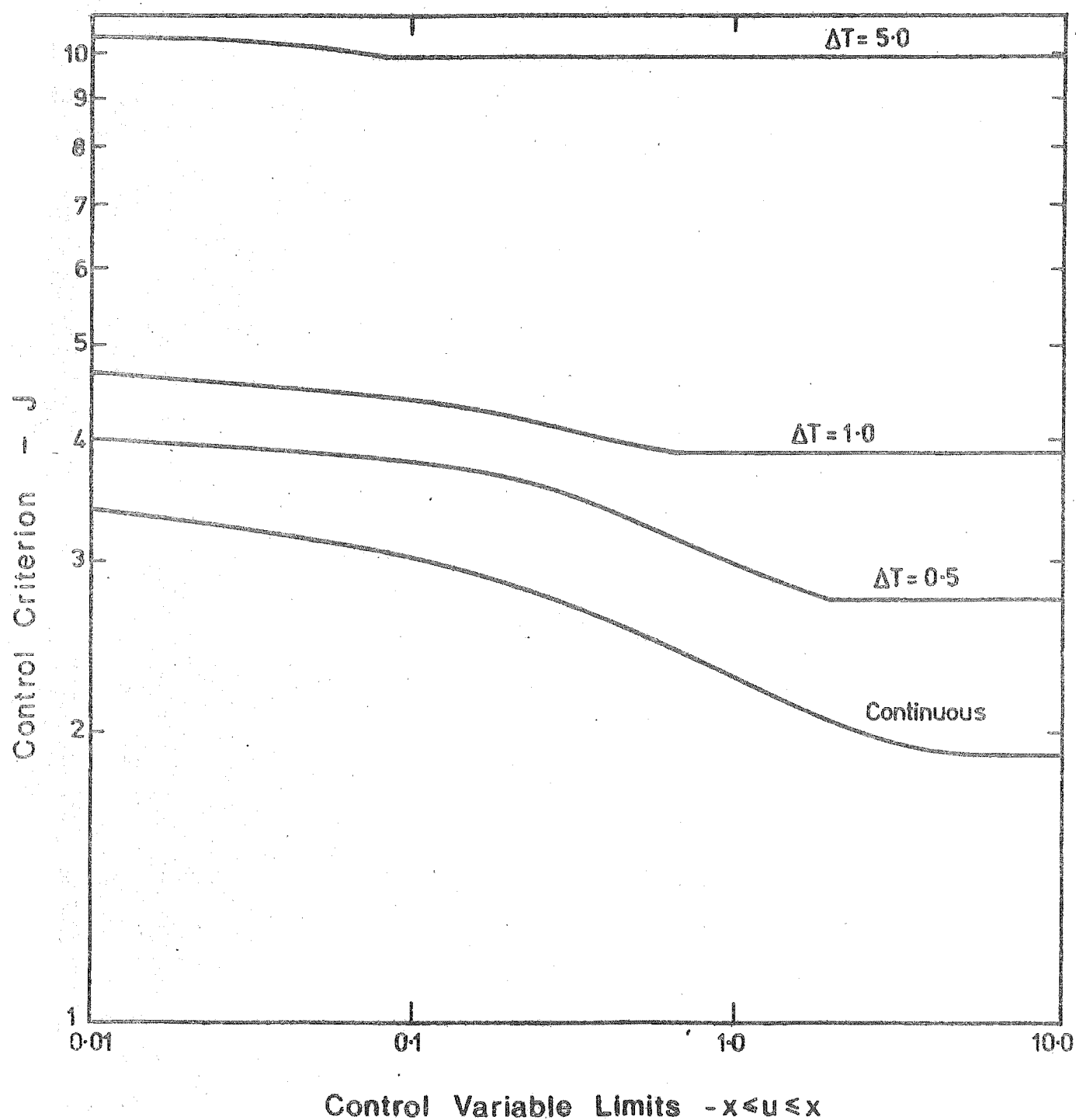


FIGURE 5.8 THE EFFECT OF REDUCED CONTROL VARIABLE LIMITS ON THE PROCESS CONTROL CRITERION FOR SYSTEM "A".

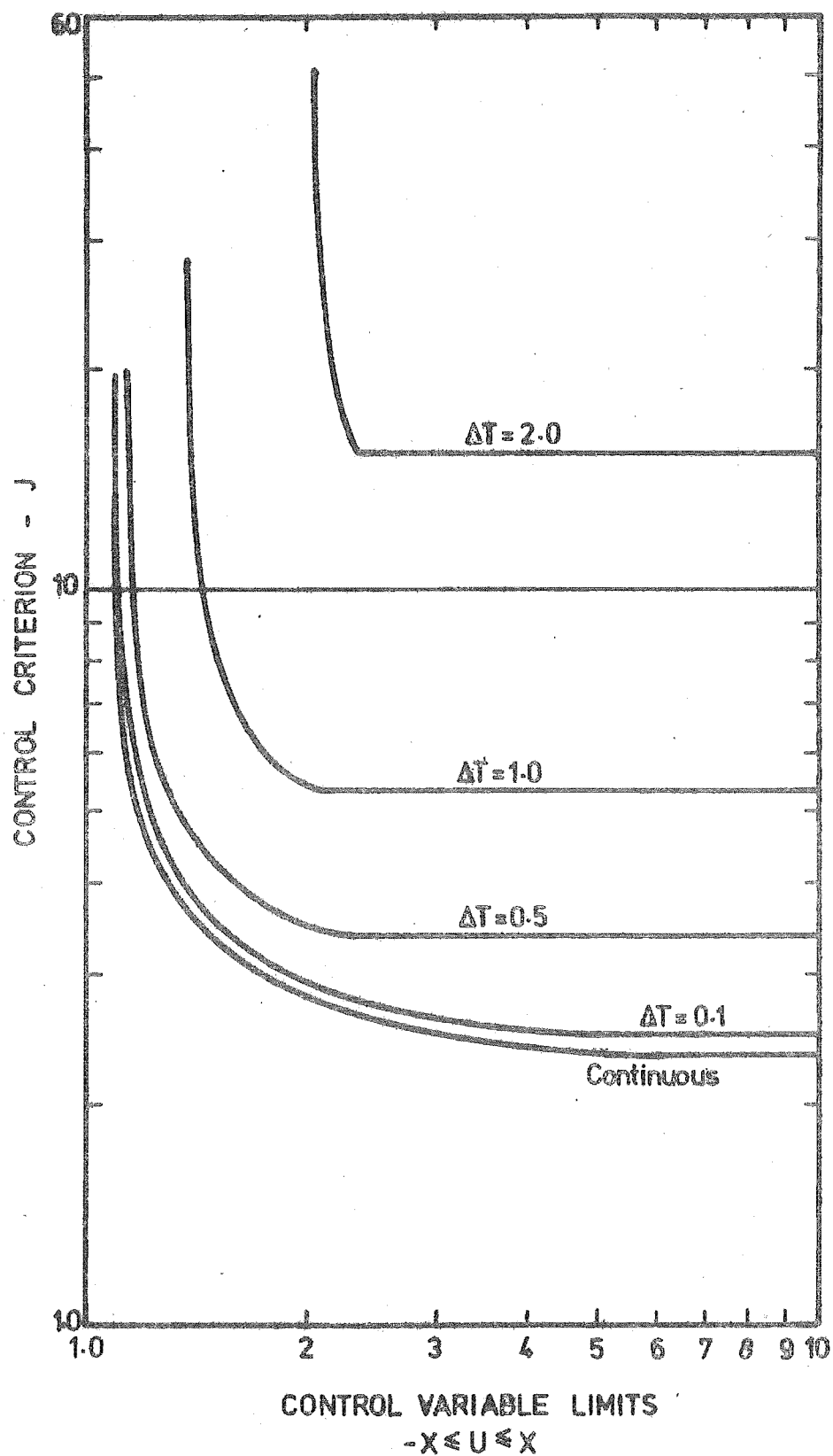


FIGURE 5.9 The degradation in quality of control, indicated by the increasing control criterion, as the control variable limits are reduced for system "B".

pronounced effect. The "optimal" feedback matrix remains optimal to within 1.0 per cent even when the control action is limited to around 10 to 20 per cent of the unbounded value.

Bounded control variables have a more dramatic effect on the unstable system "B". We have already shown that, despite increased control variable penalties, the magnitude of the feedback matrix will not decrease below a certain level, indicating the minimum level of control necessary to maintain process stability. With the control penalty maintained at  $R = 0.1$  the control limits were decreased and the effects on the process control criterion and the process stability were studied. Figure 5.9 shows the effect of decreasing control variable bounds for system "B" under continuous control and for a range of discrete intervals. Following each curve from right to left the criterion remains constant until the control variable begins to saturate against the decreasing limits. The system performance then deteriorates, slowly for the continuous model and the discrete models with short sampling intervals but more abruptly for models with longer sampling intervals, until the abscissa at the end of each line indicates the control variable limit where the system became uncontrollable.

Table 5.5 shows the maximum amount of control action used by the optimal controller in controlling model "B" from an initial state of  $x^T = [1.0 \ 1.0]$ . Also shown are the minimum control limits under which it is possible to control the process using the feedback matrix which is optimal for unbounded control.

TABLE 5.5

The variation of minimum control variable bounds with increasing discrete interval, using system "B", state and control penalty matrices  $Q = \text{diag}[1.0 \ 1.0]$ ,  $R = 0.1$ , and an initial state  $x^T = [1.0 \ 1.0]$ .

Discrete interval T	Maximum control action specified by the optimal feedback matrix.	Minimum control variable bounds necessary for stability.
Continuous	7.27	1.17
0.03	6.8	1.17
0.05	6.5	1.17
0.1	5.7	1.17
0.3	3.7	1.19
0.5	2.4	1.21
1.0	2.1	1.38
2.0	2.4	2.05
3.0	6.0	5.62
3.5	36.3	35.2
3.8	32.1	31.4
5.0	10.5	10.4
6.0	19.5	19.5

For the continuous model and the discrete models with short intervals the minimum control bounds vary little. As the sampling interval increases past a value of  $T = 1.0$  the minimum control bounds increase sharply and approach more closely the maximum values of the unbounded controls. Close to the "critical" discrete interval of  $T = 3.65$  the maximum control action and the minimum bounds on the control action are both very large in relation to the size of the process upset.

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## 5.6 SENSITIVITY OF THE SYSTEM TO PERTURBATIONS OF THE FEEDBACK MATRIX.

System "A", in closed-loop form, is relatively insensitive to perturbations of the feedback matrix coefficients and as the discrete intervals increase the sensitivity decreases. System "B" is also insensitive for small values of  $T$ , but the sensitivity increases with increasing discrete interval and with the successive reduction in control variable limits. At the point where the reduction in control action induces instability in the closed-loop response the system is highly sensitive to perturbations of the feedback matrix, which at this stage is no longer optimal.

Using system "B" as an example with a discrete interval of  $T = 5.0$  the "optimal" feedback matrix is

$$K_{FB} = \begin{bmatrix} -0.9937 & 0.7716 \end{bmatrix} \text{ based on the standard penalty}$$

matrices

$$Q = \text{diag} \begin{bmatrix} 1.0 & 1.0 \end{bmatrix} \text{ and } R = 0.1 .$$

With an initial state of  $x^T = \begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$  the maximum unconstrained control action is  $u = 10.51$ . Using the same feedback matrix the control limits were able to be reduced to  $-10.44 < u < 10.44$ , at which level the process was only just able to be stabilised. In testing the feedback matrix for optimality by perturbing the coefficients the matrix was found to be sub-optimal with respect to one and two per cent perturbations, but optimal with respect to five per cent perturbations, indicating the presence of an optimising matrix in the near vicinity of the current feedback matrix. A simple search procedure was used to locate the feedback matrix which would stabilise the system, while at the same time reducing the control variable limits. The result of this rather unsophisticated approach was that a feedback control matrix was found,

$$K_{FB} = \begin{bmatrix} -0.9940 & 0.7234 \end{bmatrix} , \text{ which could stabilise the}$$

process with control bounds of  $-10.04 < u < 10.04$ ; a considerable improvement over the "Optimal" feedback matrix. Without a nearly-optimal matrix to begin the search, however, the location of the controlling matrix and the minimum control limits would be time-consuming and costly, even for low-order systems.

## 5.7 COMPARISON OF MULTIVARIABLE AND SINGLE-LOOP CONTROLLERS.

System "B" was used to compare the performance of multivariable and single-loop proportional controllers. The single-loop controllers were implemented by having a single coefficient in the feedback matrix. Tables 5.6 and 5.7 show the important comparisons for discrete intervals of  $T = 1.0$  and  $T = 0.1$  respectively. The "Optimal" single-loop controller gain was determined by a direct search using the same process control criterion as the multivariable controller. The value of the process criterion was established over a time span of 20.0, long enough to ensure that stability had been achieved. The process criteria for the single-loop controlled processes were significantly greater than those of the multivariable controllers, and very little of the increases were due to the greater control effort used by the single-loop controllers. The final states of the multivariable controlled processes were several orders of magnitude closer to the origin than those of the single-loop controlled processes.

The introduction of control bounds demonstrates the sensitivity of the single-loop controller. The multivariable controller, using a feedback matrix designed for unbounded control, was able to stabilise the process from an initial state of  $x^T = [1.0 \ 1.0]$  with minimum control bounds of  $-1.38 < u < 1.38$ . The single-loop controller, under similar circumstances lost control of the process if the control bounds were reduced below  $-2.55 < u < 2.55$ .

A considerable reduction in the control variable limits may be made if the feedback controller gain is adjusted to accommodate the bounded control variable, but the multivariable controller still out-performs its single-loop partner by a significant margin.



TABLE 5.6

Comparison of multivariable and Single-loop controllers using process model "B" and a discrete interval  $T = 1.0$ .

No constraints on the control variable.	Multivariable control	Single-loop control
"Optimal" feedback matrix.	$\begin{bmatrix} -0.571 & 1.479 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 1.625 \end{bmatrix}$
Process criterion	5.231	9.039
Norm of state variable at $t = 20.0$	$5.18 \times 10^{-13}$	$6.82 \times 10^{-5}$
Maximum absolute value of the control variable.	2.12	3.27
Bounded control variable		
Minimum control limits for stability using "Optimal" matrix.	1.38	2.55
Feedback matrix adjusted for control variable bounds.	$\begin{bmatrix} -0.730 & 1.216 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 1.248 \end{bmatrix}$
Minimum control variable limits for stability using feedback matrix adjusted for control constraints.	1.143	1.830

TABLE 5.7

Comparison of multivariable and single-loop controllers using process model "B" and a discrete interval of  $T = 0.1$ .

No constraints on the control variable	Multivariable Control	Single-loop Control
"Optimal" feedback matrix	$\begin{bmatrix} 1.591 & 4.183 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 2.877 \end{bmatrix}$
Process Criterion	2.492	3.281
Norm of state variable at $t = 20.0$	$1.03 \times 10^{-10}$	$8.03 \times 10^{-8}$
Maximum absolute value of control variable.	5.774	2.877
Bounded control variable.		
Minimum control limits for stability using "Optimal" matrix.	1.165	1.320
Feedback matrix adjusted for control bounds.	$\begin{bmatrix} -0.40 & 6.50 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 7.325 \end{bmatrix}$
Minimum control limits for stability using feedback matrix adjusted for control bounds.	0.975	0.996

## 5.9 LIMITATION OF CONTROL ACTION BY THE CONTROL PENALTY MATRIX.

As mentioned earlier in this section the control penalty matrix may be manipulated to prevent control variable saturation. For chemical processes there would seem to be little or no justification for this approach since so few chemical processes are linear that the arguments of maintaining linearity are spurious. The control element in most chemical processes is a valve with obvious hard constraints and no damage would ensue from the control residing at either end of its range. Of course a proportioning control valve operated continuously at the end of its range argues poor process design or poor valve sizing. Similarly, a valve which is continually "slamming" from one extreme to the other in so-called "bang-bang" mode is unlikely to enhance process stability and may also significantly shorten the life of the valve. On the other hand a valve may well be regarded as grossly oversized if no saturation occurs even in the face of extreme process upsets.

Relatively few valves in chemical plant are capable of true "bang-bang" operation since they are most frequently driven by electric or pneumatic motors of limited operating speed. Solenoid valves, which are genuine "bang-bang" control elements are frequently encountered but are restricted to relatively small sizes and low operating pressures.

In relation to process time constants of many minutes, or even hours, the valve response time of the order of a minute may appear to the process to be virtually "bang-bang". Strictly speaking however this is a misnomer since for true "bang-bang" operation the control variable is permitted to reside only at the extremes of its range and there is no proportioning facility. This type of action is clearly not suited to the steady-state regulatory control of a chemical engineering process.

Control variable saturation is very common and even desirable in a process system and it is of interest to compare the performance of systems in which the action of the control variable is limited on one hand by manipulation of the control penalty matrix and on the other by hard constraints. System "B" was considered and a control penalty of  $R = 100.0$

was chosen which resulted in a continuous feedback matrix

$$K_{FB} = \begin{bmatrix} 0.005 & 2.009 \end{bmatrix}.$$

With an initial condition of  $x^T = \begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$  the control variable only just exceeds 2.0. The response of this controller was compared in Figure 5.10 with the controller based on a control penalty of  $R = 0.0001$  but with hard constraints on the control variable of  $-2.0 \leq u \leq 2.0$ . This controller provides a generally superior performance with a fast response yet critical damping and a reduced overshoot of the output variable  $x(1)$  as indicated in Table 5.8.

TABLE 5.8

Comparison of control variable limitation by control penalty and by control constraints.

	Control variable limited by large control penalty.	Control variable limited by const- raints $-2.0 < u < 2.0$
Control penalty	$R = 100.0$	$R = 0.0001$
Feedback matrix	$\begin{bmatrix} 0.005 & 2.009 \end{bmatrix}$	$\begin{bmatrix} 99.7 & 102.0 \end{bmatrix}$
Process control Criterion (Based on an equivalent control penalty $R = 0.1$ )	3.88	3.18
Norm of the state variable at $t = 10.0$	$1.23 \times 10^{-2}$	$2.14 \times 10^{-5}$
Type of response	Slow and oscillatory.	Fast and critically damped.

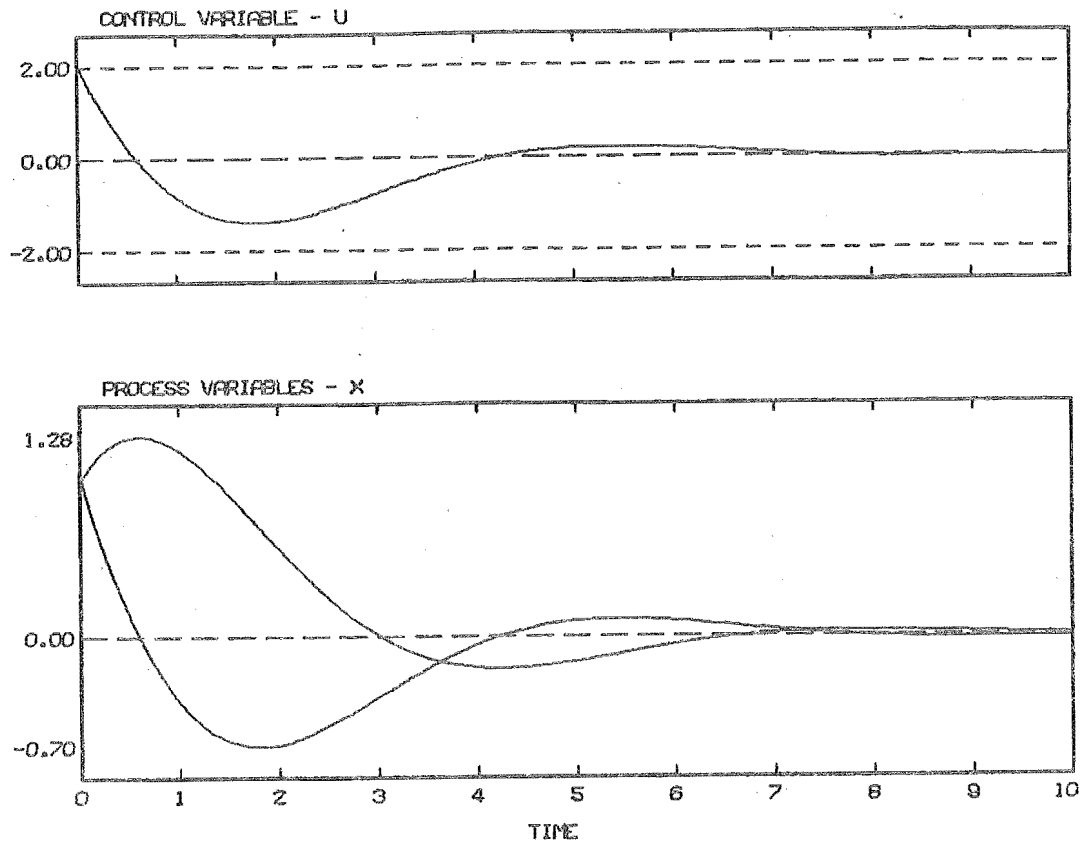


FIGURE 5.10 (A) THE EFFECT OF A HEAVY CONTROL PENALTY ( $R = 100.0$ ) TO LIMIT THE CONTROL VARIABLE, CONTROL IS SLUGGISH AND OSCILLATORY.

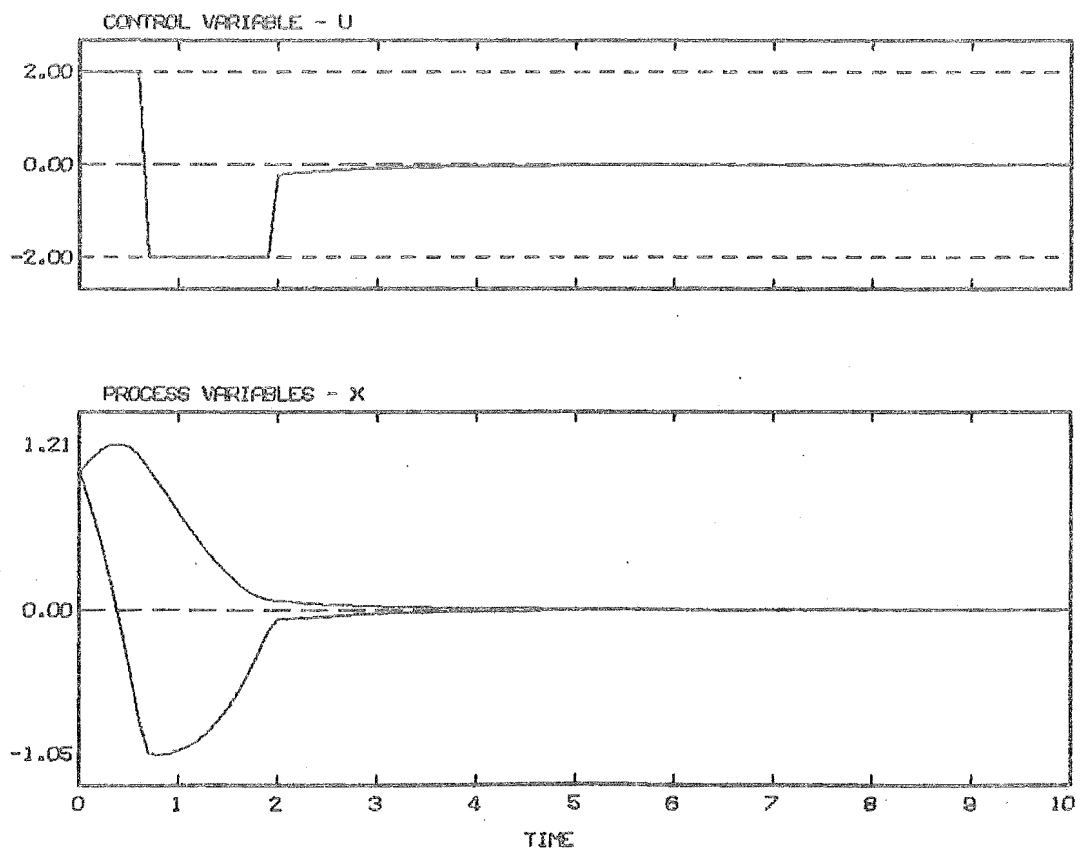


FIGURE 5.10 (B) A LIGHT CONTROL PENALTY ( $R = 0.0001$ ) BUT CONTROL VARIABLE LIMITS OF  $-2.0 < U < 2.0$ , RESULTING IN A FAST, NON-OSCILLATORY RESPONSE.

## CHAPTER VI

### CONTROL OF UNSTABLE NON-LINEAR CHEMICAL REACTORS

Two unstable chemical reactors are developed to demonstrate the techniques of process modelling and linearisation. Linear multivariable controllers are established and their performance is compared with single-loop controllers in stabilising the reactors under conditions of random disturbances, bounded control variables and process dead-time.

	Page
6.1 Development of Reactor Models	84
6.2 REACTOR-2-1	84
6.3 REACTOR-2-2	87
6.4 Numerical Parameters for the Reactor Models	89
6.5 Process and Control Variable Constraints	89
6.6 Zone of Controllability	91
6.7 Comparison of Multivariable and Multi-loop Controllers	94
6.8 Stability of REACTOR-2-1 with Random Step Disturbances	96
6.9 The Effect of Control Variable Dead-time on REACTOR-2-1	99
6.10 Stability of REACTOR-2-2 with Random Disturbances and Dead-time	99

## CHAPTER VI

## CONTROL OF NON-LINEAR STIRRED TANK REACTORS

## 6.1 DEVELOPMENT OF THE REACTOR MODELS.

The reactor used in this section is fictitious to the extent that no attempt has been made to model a real reactor or to simulate the kinetics of any particular reaction. The model has been chosen to combine simplicity with suitable non-linearity and an autocatalytic reaction mechanism is proposed to ensure instability. Two forms of the reactor are investigated, the first with two state variables and one control variable, REACTOR-2-1, the second with two of each, REACTOR-2-2.

## 6.2 REACTOR-2-1

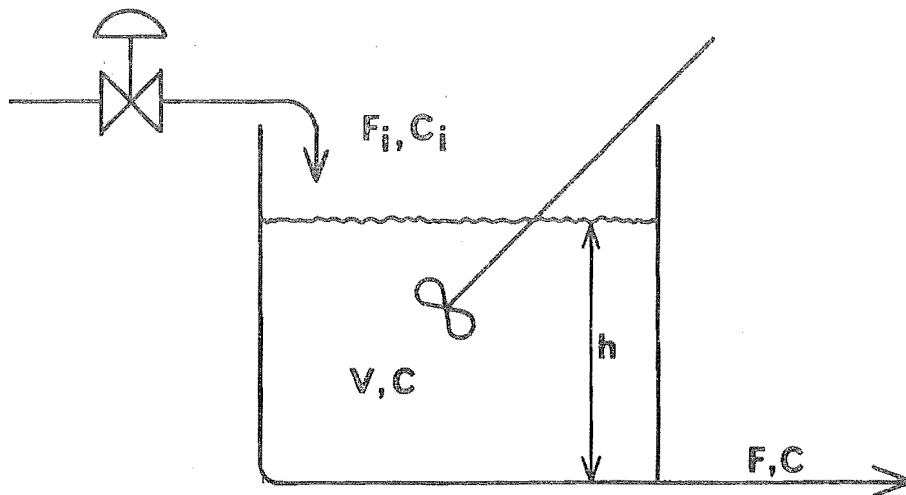


FIGURE 6.1 Continuous Stirred Tank Reactor with control only on the inlet flow rate.

The kinetics of the autocatalytic isothermal reaction are second order and the rate of reaction is

$$dC/dt = kC^2 \quad \text{.....(6.1)}$$

The flow rate into the reactor is the controlled variable but the flow from

the reactor, assuming turbulent flow, is given by

$$Q = R\sqrt{h} \quad \text{.....(6.2)}$$

where  $R$  is chosen to suit the mean flow rate  $F_s$ , the mean inlet concentration and the required product concentration. Likewise the volume of reactant, and hence the level in the reactor  $h$ , is governed by the flow rate and concentration. The dynamic behaviour of the reactor may be determined from total and component mass balances. The total mass balance over the reactor gives

$$\rho \frac{dV}{dt} = \rho F - \rho Q \quad \text{.....(6.3)}$$

where  $\rho$  is the density of the reactant and is assumed constant. Substituting  $Q$  from Equation 6.2 and using  $V = Ah$ , where  $A$  is the cross-sectional area of the reactor

$$\frac{dh}{dt} = \frac{F}{A} - \frac{R\sqrt{h}}{A} \quad \text{.....(6.4)}$$

The component mass balance on the reacting species gives

$$\frac{d}{dt}(VC) = FC_I - QC + VKC^2 \quad \text{.....(6.5)}$$

and combining with equation 6.1 gives

$$\frac{dC}{dt} = \frac{F}{Ah}(C_I - C) + KC^2 \quad \text{.....(6.6)}$$

and equations 6.4 and 6.6 are the non-linear state equations governing the behaviour of the reactor. The inlet concentration  $C_I$  is variable but has a mean value of  $C_{IS}$ , and the level of reactant corresponding to the required reactor volume is  $h_s$ , given by

$$h_s = F_s(C_s - C_{IS})/AKC_s^2 \quad \text{.....(6.7)}$$

The required resistance of the outlet valve may be calculated from

$$R = F_s/\sqrt{h_s} \quad \text{.....(6.8)}$$

to ensure that at the mean inlet concentration and steady-state flow rate the reactor volume is correct.

Dimensionless deviation variables are chosen such that



$$x_1 = (h - h_s)/h_s$$

$$x_2 = (C - C_s)/C_s$$

The controlled variable is the inlet flow rate

$u = (F - F_s)/F_s$  and the disturbance variable is the fluctuation in inlet concentration

$d = (C_I - C_{Is})/C_{Is}$  Using these deviation variables the non-linear state equations become

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x, u, d) \\ &= F_s \left[ (1 + u) - (1 + x_1)^{1/2} \right] / Ah_s \end{aligned} \quad \dots\dots(6.9)$$

$$\begin{aligned} \frac{dx_2}{dt} &= f_2(x, u, d) \\ &= \frac{F_s}{Ah_s} \left[ \frac{1+u}{1+x_1} \right] \left[ \frac{C_{Is}}{C_s} (1+d) - (1+x_2) \right] \\ &\quad + \frac{F_s}{Ah_s} \left[ \frac{C_s - C_{Is}}{C_s} \right] (1+x_2)^2 \end{aligned} \quad \dots\dots(6.10)$$

In order to apply the linear control theory it is necessary to linearise the process and this is accomplished by forming the Jacobean matrices  $A = \partial f_i / \partial x_j$ ,  $B = \partial f_i / \partial u_j$ , and  $C = \partial f_i / \partial d_j$ . The elements of the A, B and C matrices simplify considerably if the reactor can be assumed to operate at or about its steady state of

$$x_1 = x_2 = u = d = 0.$$

The elements of the matrices in the linear steady-state model are given by

$$\begin{aligned} a_{11} &= \partial f_1 / \partial x_1 = -0.5 F_s / Ah_s \\ a_{12} &= \partial f_1 / \partial x_2 = 0.0 \\ a_{21} &= \partial f_2 / \partial x_1 = F_s (C_s - C_{Is}) / Ah_s C_s \\ a_{22} &= \partial f_2 / \partial x_2 = F_s (C_s - 2C_{Is}) / Ah_s C_s \\ b_{11} &= \partial f_1 / \partial u = F_s / Ah_s \\ b_{21} &= \partial f_2 / \partial u = -F_s (C_s - C_{Is}) / Ah_s C_s \\ c_{11} &= \partial f_1 / \partial d = 0.0 \\ c_{21} &= \partial f_2 / \partial d = F_s C_{Is} / Ah_s C_s \end{aligned}$$

The reactor has eigenvalues  $-F_s/Ah_s$  and  $F_s(C_s - 2C_{Is})/Ah_s C_s$ , and in order for the model to display the desired instability it is necessary that the steady-state outlet concentration  $C_s$  be always greater than twice the inlet concentration  $C_{Is}$ . The level of reactant in the reactor is inherently stable, although the reactor may overflow or run dry.

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### 6.3 REACTOR-2-2

The second reactor is chosen to be the same in all respects except that the valve at the reactor exit is also controlled.

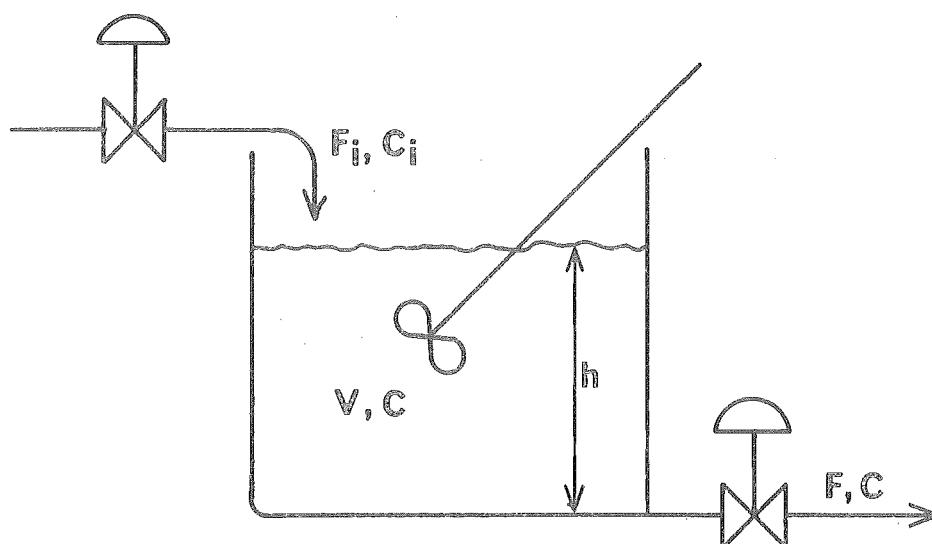


FIGURE 6.2 Continuous Stirred Tank Reactor with control of both inlet and outlet flow rates.

A total mass balance over the reactor gives

$$\rho dV/dt = \rho F_i - \rho F \quad \text{and using } V = Ah \text{ gives}$$

$$\frac{dh}{dt} = \frac{F_i}{A} - \frac{F}{A} \quad \text{.....(6.11)}$$

a component mass balance for the reacting species provides

$$d(CV)/dt = F_i C_i - FC + VkC^2 \quad \text{which simplifies to}$$

$$\frac{dC}{dt} = F_I (C_I - C)/Ah + kC^2 \quad \text{.....(6.12)}$$

Defining the dimensionless deviation variables as before but with

$$u_1 = (F_I - F_s)/F_s$$

and  $u_2 = (F - F_s)/F_s$  results in the non-linear state equations:

$$\begin{aligned} \frac{dx_1}{dt} &= f_1(x, u, d) \\ &= F_s u_1 / Ah_s - F_s u_2 / Ah_s \quad \text{.....(6.13)} \end{aligned}$$

$$\begin{aligned} \frac{dx_2}{dt} &= f_2(x, u, d) \\ &= \frac{F_s}{Ah_s} \left[ \frac{1 + u_1}{1 + x_1} \right] \left[ \frac{C_{Is}}{C_s} (1 + d) - (1 + x_2) \right] \\ &\quad + \frac{F_s}{Ah_s} \left[ \frac{C_s - C_{Is}}{C_s} \right] (1 + x_2)^2 \quad \text{.....(6.14)} \end{aligned}$$

Linearisation produces a steady-state linear model with the elements of the A, B and C matrices given by

$$\begin{aligned} a_{11} &= \partial f_1 / \partial x_1 = 0.0 \\ a_{12} &= \partial f_1 / \partial x_2 = 0.0 \\ a_{21} &= \partial f_2 / \partial x_1 = F_s (C_s - C_{Is}) / Ah_s C_s \\ a_{22} &= \partial f_2 / \partial x_2 = F_s (C_s - 2C_{Is}) / Ah_s C_s \\ b_{11} &= \partial f_1 / \partial u_1 = F_s / Ah_s \\ b_{12} &= \partial f_1 / \partial u_2 = -F_s / Ah_s \\ b_{21} &= \partial f_2 / \partial u_1 = F_s C_{Is} / Ah_s C_s \\ b_{22} &= \partial f_2 / \partial u_2 = -F_s / Ah_s \\ c_{11} &= \partial f_1 / \partial d = 0.0 \\ c_{21} &= \partial f_2 / \partial d = F_s C_{Is} / Ah_s C_s \end{aligned}$$

The conditions necessary to promote instability are the same as the previous example, but instead of one negative eigenvalue, indicating a stable mode, there is now a zero eigenvalue indicating a purely integrating effect. Because of the two control variables the evaluation of the control algorithm is necessarily more complex but it would be reasonable to expect more effective control.

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#### 6.4 NUMERICAL PARAMETERS FOR THE REACTOR MODELS.

Simple numbers were chosen for the process parameters to avoid unnecessary collections of numerals. Process instability may be assured, as indicated above, with

$$C_{Is} = 1.0 \times 10^3$$

and  $C_s = 1.0 \times 10^4$

With a reaction rate constant

$$k = 9.0 \times 10^{-5} \text{ and a steady-state}$$

flow rate of

$F_s = 1.0$  the reactor is found, fortuitously, to require a volume  $V = 1.0$  resulting in a reactor space-time  $\tau = 1.0$

The linear steady-state model for REACTOR-2-1 is therefore

$$\dot{\bar{x}} = \begin{bmatrix} -0.5 & 0.0 \\ 0.9 & 0.8 \end{bmatrix} \bar{x} + \begin{bmatrix} 1.0 \\ -0.9 \end{bmatrix} u + \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix} d \quad \dots\dots(6.15)$$

and for REACTOR-2-2

$$\dot{\bar{x}} = \begin{bmatrix} 0.0 & 0.0 \\ 0.9 & 0.8 \end{bmatrix} \bar{x} + \begin{bmatrix} 1.0 & -1.0 \\ 0.1 & -1.0 \end{bmatrix} u + \begin{bmatrix} 0.0 \\ 0.1 \end{bmatrix} d \quad \dots\dots(6.16)$$

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#### 6.5 PROCESS AND CONTROL VARIABLE CONSTRAINTS.

The ability of the linear controller to return the process to the designed steady-state from an arbitrary initial condition in the state space was investigated using both continuous and discrete controllers. The natural control variable bounds for both reactor models were  $-1.0 \leq u \leq 1.0$ , corresponding to the control valves operating at their mid-points under steady-state conditions. In addition, however, there were

constraints on the state variables resulting from the physical models used; for example, the level of reactant in the reactor could not be less than zero, implying  $x_1 \geq -1.0$ . The maximum allowable level in the reactor was chosen to be 50% above the designed steady-state level, i.e.  $x_1 \leq 0.5$ . This posed some minor problems in determining the allowable range of the control variables. For REACTOR-2-1 it was necessary only to specify to the continuous controller that if the level was approaching the top of the reactor, i.e.  $x_1$  approaching 0.5, the inlet flow be limited to the rate at which the reactant could be drained away. If a steady-state situation is envisaged at  $x_1 = 0.5$ , then from equation 6.8, and using the chosen numerical parameters

$$(1 + u) = (1 + 0.5)^{1/2} \text{ and thus } u = 0.224$$

For the discrete controller the situation is a little more complex. The controller is required to specify at the beginning of the control period a value of the control variable such that the state variable constraints will not be violated at any stage of the control period. To prevent overflowing the reactor this is equivalent to specifying that the current level in the reactor plus the product of the rate of increase in the level and the discrete time interval should not exceed 0.5, i.e.

$$x_1 + \frac{dx_1}{dt}T \leq 0.5$$

Since  $\frac{dx_1}{dt} = F_s [(1 + u) - (1 + x_1)^{1/2}] / Ah_s$ , this results in

$$u \leq Ah_s(0.5 - x_1) / TF_s + (1 + x_1)^{1/2} \quad \dots\dots(6.17)$$

For REACTOR-2-2 under continuous control it is necessary to specify only that  $u_1 \leq u_2$  when  $x_1 \geq 0.5$ . Under discrete time control the requirement becomes

$$x_1(k+1) \leq 0.5$$

$$= \phi_{11}x_1(k) + \delta_{11}u_1(k) + \delta_{12}u_2(k) \quad \text{where the } \phi \text{ and } \delta$$

are elements of the transition and driving matrices,  $\phi$  and  $\Delta$ . Since the control variables are already bounded the inlet flow gains an additional upper constraint;

$$u_1(k) \leq [0.5 - \phi_{11}x_1(k) - \delta_{12}u_2(k)] / \delta_{11}$$

The implementation of additional overriding control constraints is trivial if a digital control computer is available, but would be very difficult in the case of an analogue controller. It is important to note, however, that these additional constraints are of no assistance to the performance of the linear control algorithm.

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#### 6.6 ZONE OF CONTROLLABILITY.

As a means of comparing the performance of continuous and discrete controllers for both reactors a zone of controllability was determined. The perimeter of the zone was determined by trial and error such that if the process state was within the zone the controller could return the process to the desired steady-state in a finite time. Only one discrete time interval was investigated because establishing the perimeter was a very time-consuming and expensive process. A discrete-time interval of  $T=0.5$  was used and when this is compared to the reactor steady-state space time of 1.0 the quality of the control would be expected to suffer as a result. For most industrial installations the control interval would probably be less than one tenth of the dominant time constant or mean residence time. For unstable processes the control interval might be much shorter.

Figure 6.3 shows the controllability zones for REACTOR-2-1 and the expected degradation in performance from continuous to discrete controller is confirmed. There is a large uncontrollable sector under conditions of high concentration and high reactor level which is due to the auto-catalytic reaction. In order to control the runaway reaction the controller attempts to dilute and flush away the high concentration of reactant but it is limited by the rate at which the reactant drains from the reactor and by the upper limit imposed on the level of reactant in the vessel, which is why the combination of high concentration and high level is potentially uncontrollable. Because of the free-draining nature of the reactor and very low reaction rate at low concentration, due to the non-linear reaction

# REGION OF CONTROLLABILITY

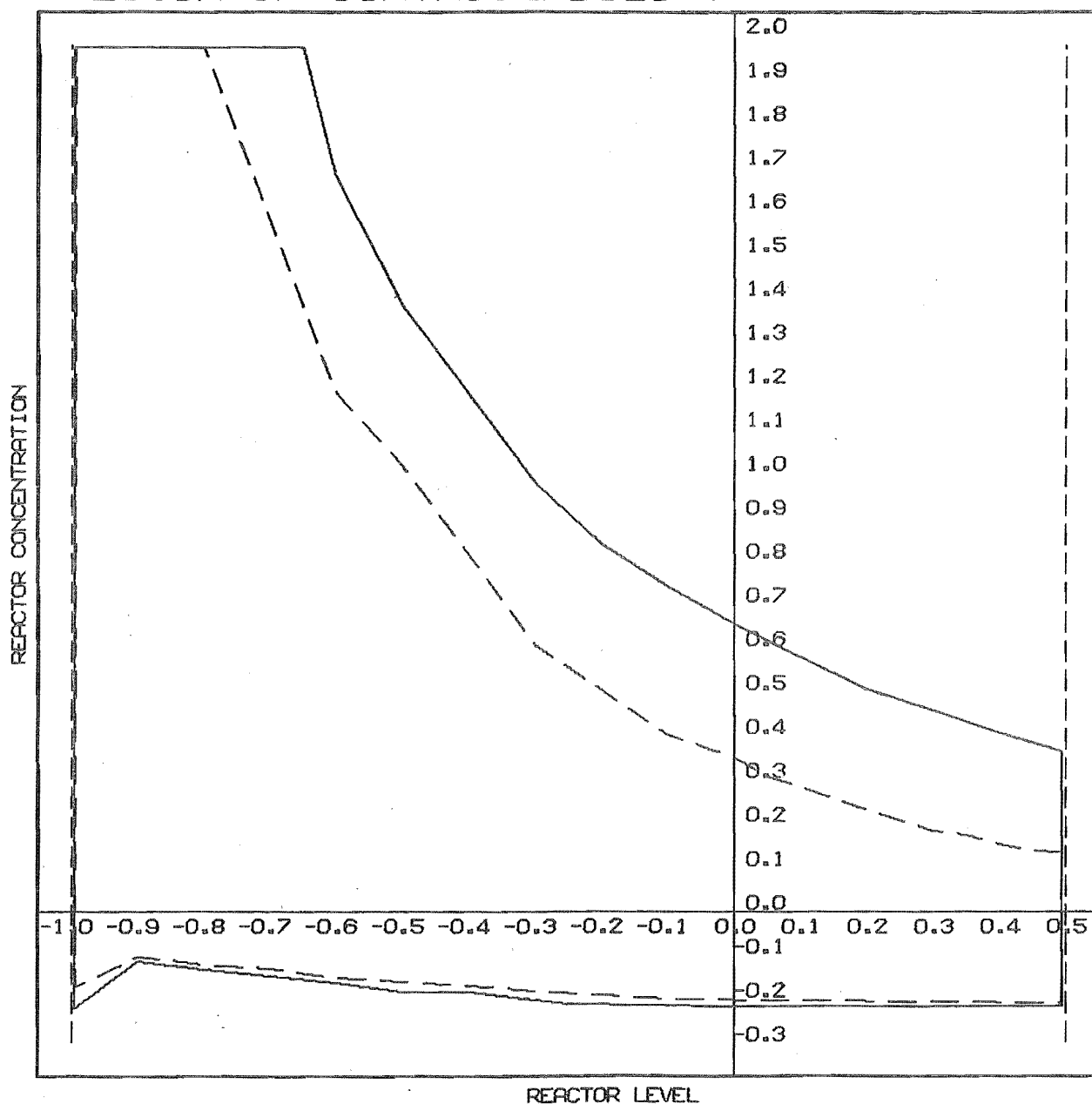


FIGURE 6.3 ZONE OF CONTROLLABILITY FOR REACTOR-2-1 SHOWING UNCONTROLLABLE REGION AT HIGH CONCENTRATION AND REACTOR LEVEL. CONTROLLABLE REGION FOR DISCRETE CONTROLLER ( $T = 0.5$ ) IS CONSIDERABLY REDUCED.

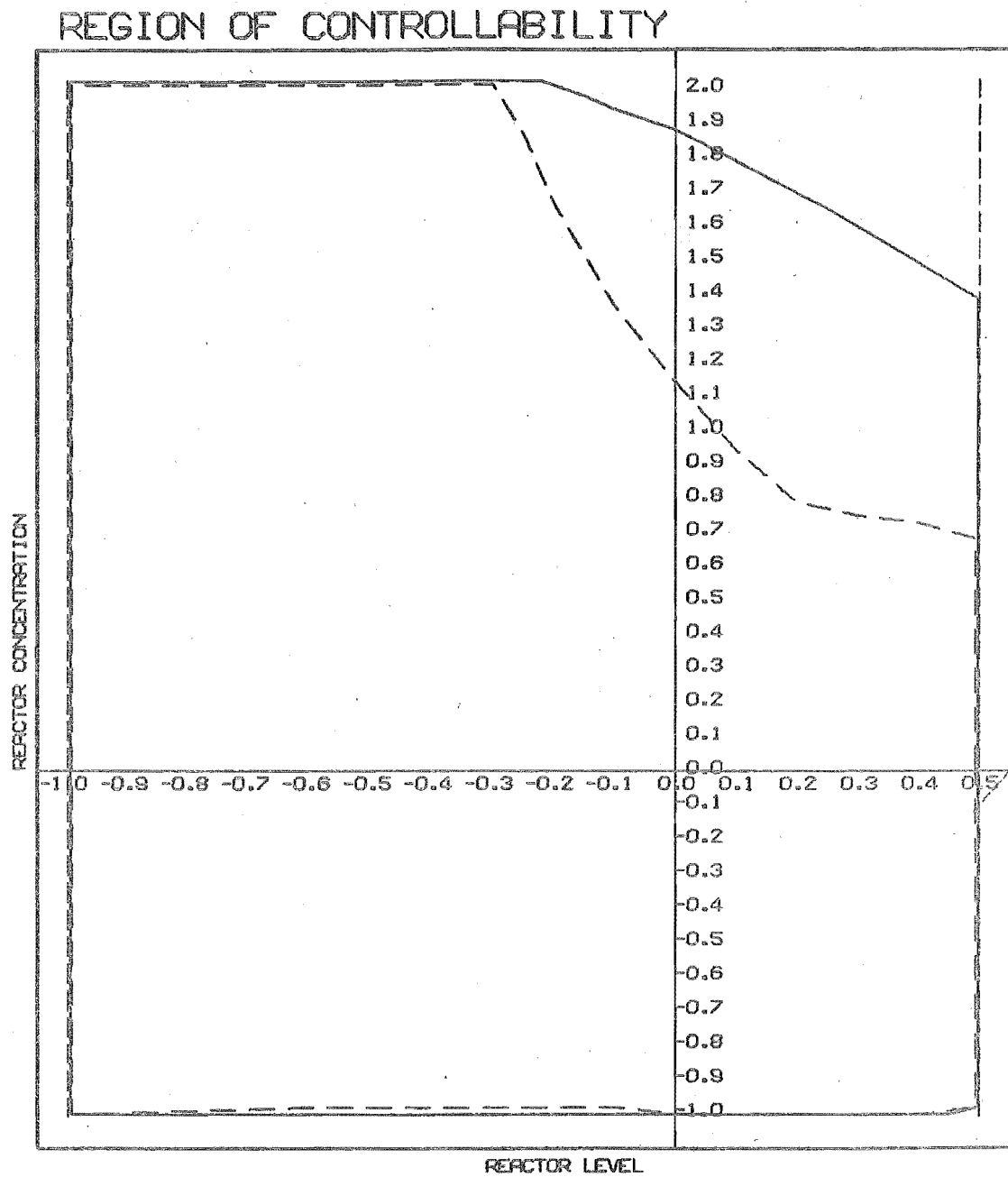


FIGURE 6.4 ZONE OF CONTROLLABILITY FOR REACTOR-2-2 SHOWING CONTROLLER'S ABILITY TO RECOVER FROM CONDITIONS OF VERY LOW CONCENTRATION.



kinetics  $dC/dt = kC^2$ , the reaction is not self-sustaining below an initial concentration of  $C = 7\ 800$  corresponding to  $x_2 = -0.22$ .

Clearly, in order to get the reaction started, it would be necessary to prevent the reactor from emptying itself entirely.

The dramatically increased zone of controllability for REACTOR-2-2 is shown in Figure 6.4. The uncontrollable sector at high concentration and level of reactants is considerably reduced, although there is still a marked difference in performance between the continuous and discrete controllers. A major improvement due to the additional control variable is the controller's ability to recover from conditions of very low reactant concentration simply by closing both control valves and waiting for the concentration and thus the reaction rate to increase. Needless to say, the recovery rate from such low concentration conditions was very slow.

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#### 6.7 COMPARISON OF MULTIVARIABLE AND MULTI-LOOP CONTROLLERS.

Multi-Loop controllers were found for REACTORS-2-1 and 2-2 by an iterative search minimising the same control criterion as the multivariable controllers. The controller feedback gain matrices are shown in Table 6.1 and their performance on the linearised process models is compared. The advantage of the multivariable controller is considerable for REACTOR-2-1, but much less so for REACTOR-2-2. The multivariable feedback coefficient  $k_{12}$  was dominant for REACTOR-2-1, because of the heavy penalty applied to concentration deviations, so it was not surprising to find the single-loop controller with a feedback gain of very similar magnitude.

The configuration problem for REACTOR-2-2 was also simplified by the dominant element  $k_{22}$ , in the multivariable feedback matrix. The most likely pair were thus  $k_{11}$  and  $k_{22}$  which meant that the inlet flow-rate of reactant was controlled by the level in the reactor and the outlet flow-rate was

controlled by the concentration of the product. It was not possible to find a single-loop discrete controller for REACTOR-2-1 with a discrete interval of  $T=0.5$  despite an extensive search.

TABLE 6.1

Comparison of the performance of multivariable and single-loop controllers on linear models of REACTOR-2-1 and REACTOR-2-2.

REACTOR-2-1 with continuous controller.	Multivariable controller	Single-loop controller
Feedback Matrix	$\begin{bmatrix} 3.443 & 39.141 \end{bmatrix}$	$\begin{bmatrix} 0.0 & 38.005 \end{bmatrix}$
Control Criterion for an initial state $x(0)^T = \begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	34.7	47.2
Norm of state at time $t = 20.0$	$1.96 \times 10^{-8}$	$1.58 \times 10^{-4}$
REACTOR-2-2 with Discrete controller ( $T = 0.5$ )	Multivariable Controller	Multi-loop Controller
Feedback Matrix	$\begin{bmatrix} 1.098 & -1.449 \\ -0.464 & -2.368 \end{bmatrix}$	$\begin{bmatrix} 2.589 & 0.0 \\ 0.0 & -2.043 \end{bmatrix}$
Control Criterion for an initial state of $x(0)^T = \begin{bmatrix} 1.0 & 1.0 \end{bmatrix}$	50.94	52.99
Norm of state variable at time $t = 20.0$	$8.70 \times 10^{-14}$	$6.84 \times 10^{-12}$

## 6.8 STABILITY OF REACTOR-2-1 WITH RANDOM STEP DISTURBANCES.

A reasonable test for these controllers designed specifically for regulatory purposes was their ability to minimise process variable deviations in the face of load variable upsets. The load variable was chosen to be the inlet concentration and the controllers were tested by subjecting the process to a sequence of random step changes above and below the design inlet concentration of  $C_{Is} = 1000$ . The minimum period of the step change was chosen to be 0.5, corresponding to half the reactor space-time and coinciding also with the control interval of the particular discrete model under investigation.

Figure 6.5 shows the control criterion, for a total control period of 20.0, plotted against the absolute value of the deviation of the disturbance variable. It will be noted from the definition of the disturbance variable

$$d = (C_I - C_{Is})/C_{Is} \quad \text{that values of } d \text{ less than } -1.0$$

are physically meaningless for this model, however deviations beyond this were plotted to determine the level of disturbance at which each system became unstable. The multivariable controller maintained much tighter control of the process with a control criterion between one half and two thirds that of the single-loop controller. Also, the single-loop controller was unable to maintain control of the process for disturbances above  $d = \pm 2.0$ , whereas the multivariable controller was still in control with deviations as large as  $d = \pm 2.5$ .

Underlining the efficacy of the multivariable controller was its ability to minimise the effect of the disturbances with less control action than the single-loop controller. While the single-loop control variable was saturating for disturbances of  $d = \pm 0.5$  the multivariable controller did not saturate until  $d = \pm 1.0$ . The quality of control achieved by the continuous single-loop controller was very similar to the control achieved by the discrete multivariable controller with a control interval of  $T=0.25$ , however the discrete controller maintained control of the process for greater disturbance amplitudes. Figure 6.5 also shows the degradation in control quality and the earlier onset of instability as the control interval was increased from  $T=0.25$  to  $T=0.5$  and  $T=1.0$ .

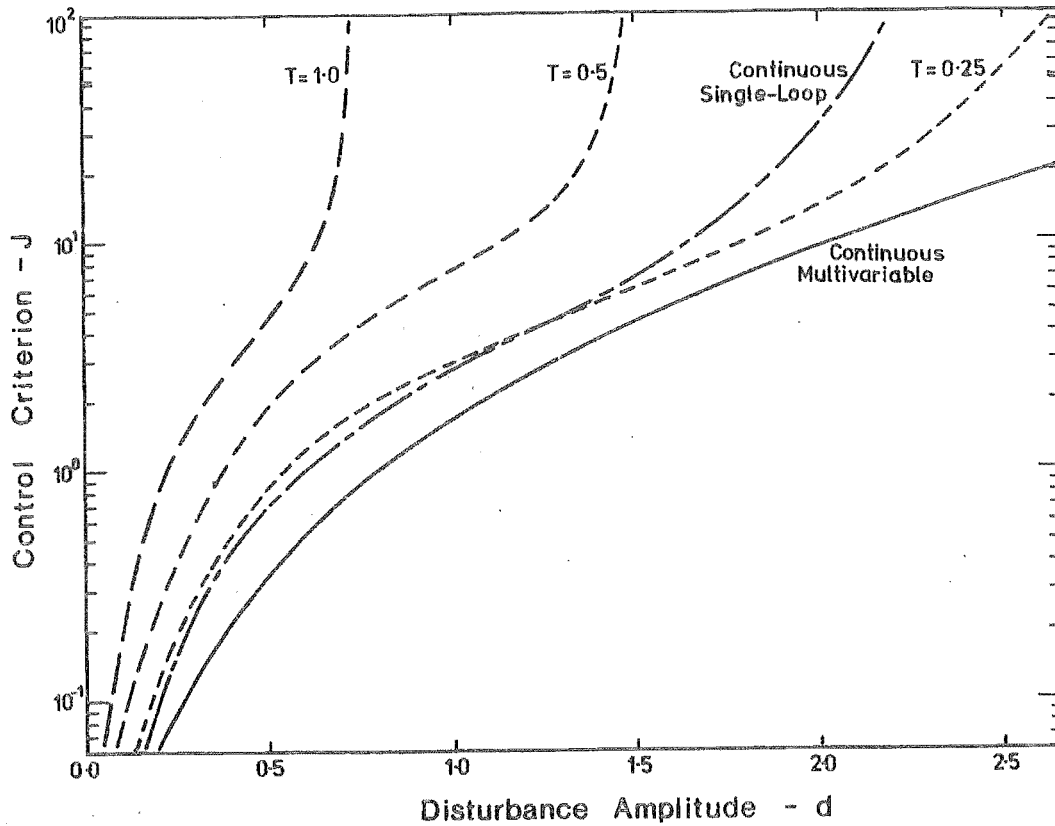


FIGURE 6.5 VARIATION OF CONTROL CRITERION WITH INCREASING AMPLITUDE OF THE DISTURBANCE VARIABLE. THE VERY RAPID DEGRADATION OF CONTROL QUALITY WITH INCREASING DISCRETE INTERVAL IS APPARENT, ALSO THE ADVANTAGE OF THE CONTINUOUS MULTIVARIABLE OVER THE CONTINUOUS SINGLE-LOOP CONTROLLER AT HIGH DISTURBANCE AMPLITUDES.

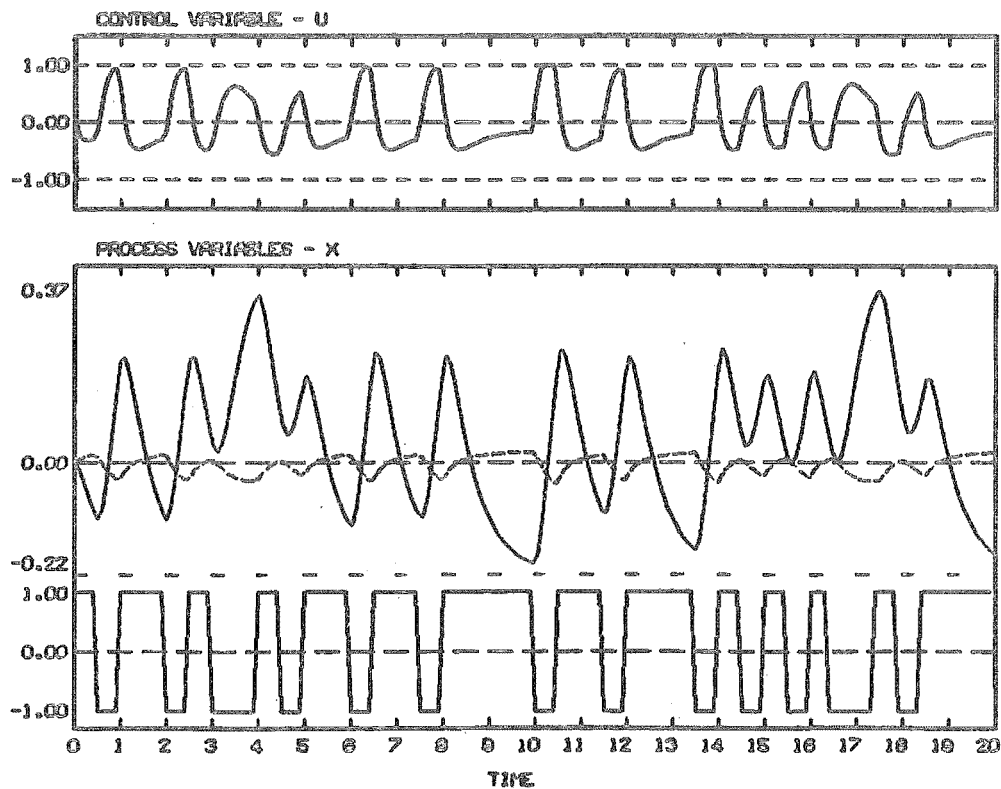


FIGURE 6.6 PROCESS VARIABLE TRAJECTORIES FOR REACTOR-2-1 UNDER CONTINUOUS MULTIVARIABLE CONTROL AND SUBJECTED TO RANDOM STEP CHANGES IN INLET CONCENTRATION OF AMPLITUDE  $\pm 1.0$ .

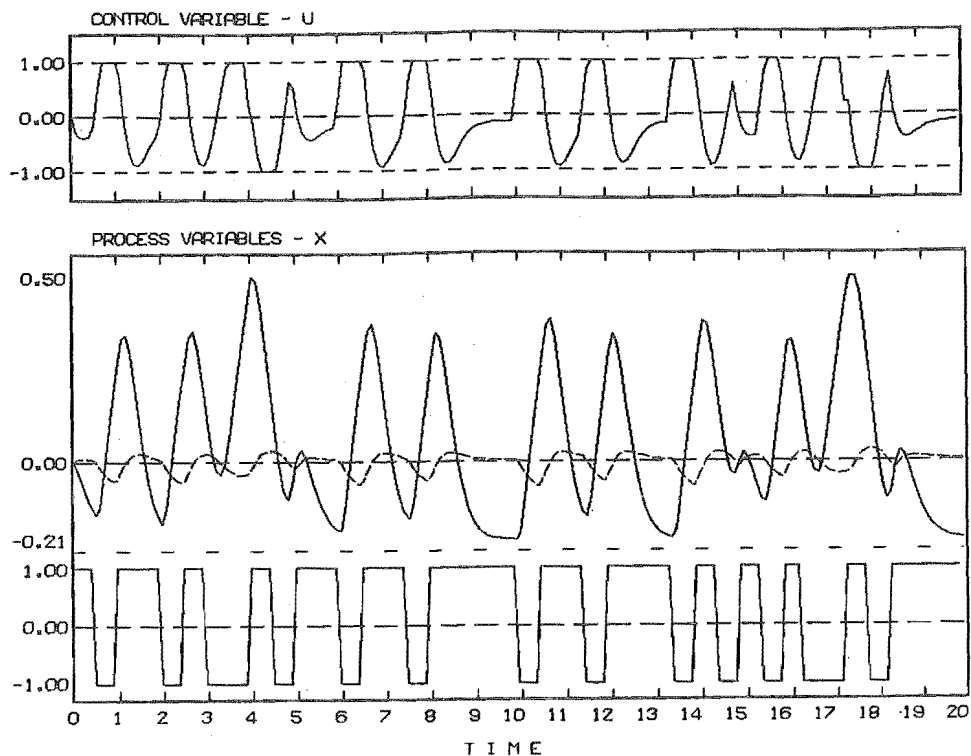


FIGURE 6.7 PROCESS VARIABLE TRAJECTORIES FOR REACTOR-2-1 UNDER CONTINUOUS SINGLE-LOOP CONTROL. NOTE THE FREQUENT SATURATION OF THE CONTROL VARIABLE AND THE LARGER DEVIATIONS OF THE PROCESS VARIABLES.

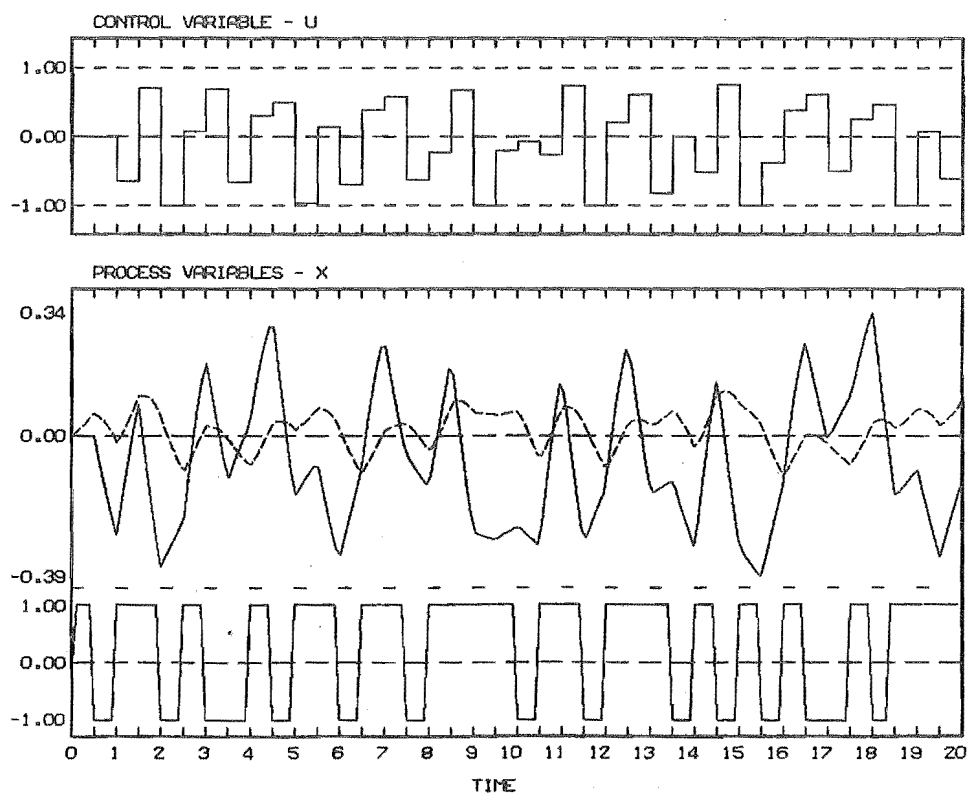


FIGURE 6.8 PROCESS VARIABLE TRAJECTORIES FOR REACTOR-2-1 UNDER DISCRETE MULTIVARIABLE CONTROL ( $T = 0.5$ ). THE CONCENTRATION DEVIATIONS (DOTTED LINE) ARE MUCH LARGER THAN EITHER OF THE CONTINUOUS CONTROLLERS.

Comparative process trajectories for a disturbance variable amplitude of  $d = \pm 1.0$  are shown in Figures 6.6, 6.7 and 6.8 for continuous multivariable, continuous single-loop and discrete multivariable ( $T=0.5$ ) controllers respectively.

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#### 6.9 THE EFFECT OF CONTROL VARIABLE DEAD-TIME ON REACTOR-2-1.

As stated in the introduction chemical processes are frequently complicated by state or control variable delays, or "dead-time". Whether it is the distance-velocity lag of material in a pipeline or the delay due to on-line or off-line analysis, dead-time poses particular problems to the process control engineer. The stability of the reactors, using controllers based on no-delay models, was evaluated using random disturbances and increasing dead-time.

The continuous multivariable controller showed a considerable margin of stability over the single-loop controller as shown in Figure 6.9.

The curves show the increase in control criterion with increasing dead-time for a given level of disturbance amplitude. The multivariable controllers, both continuous and discrete, have a definite advantage over the single-loop controller in the amount of dead-time they can tolerate.

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#### 6.10 STABILITY OF REACTOR-2-2 WITH RANDOM DISTURBANCES AND DEAD-TIME.

With two variables to manipulate the second reactor was able to maintain a tighter control over the process fluctuations, as may be seen in

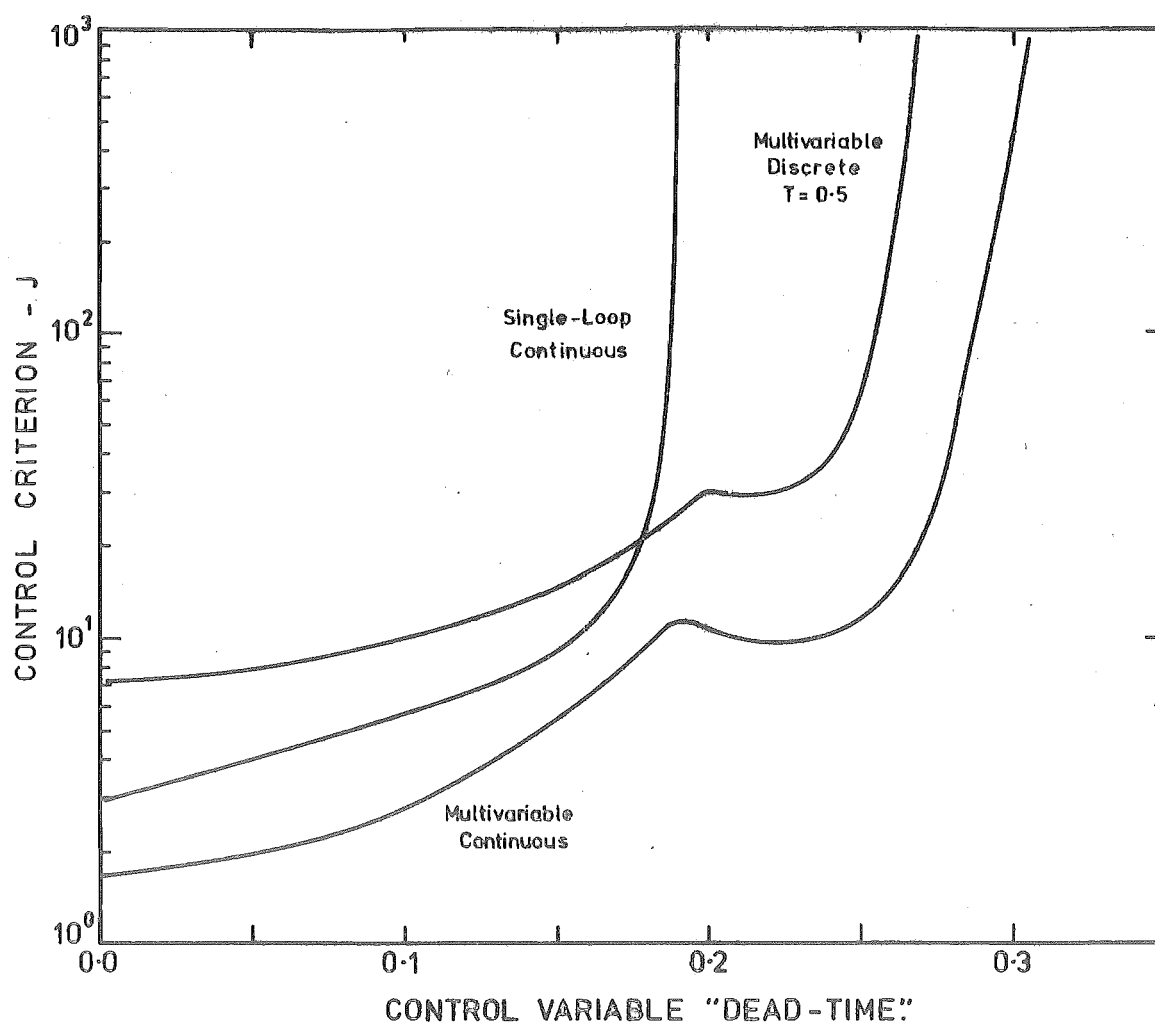


FIGURE 6.9 THE EFFECT OF INCREASING CONTROL VARIABLE "DEAD-TIME" ON REACTOR STABILITY UNDER RANDOM STEP CHANGES IN FEED CONCENTRATION. THERE IS A SIGNIFICANT MARGIN IN STABILITY FOR THE MULTIVARIABLE CONTROLLERS.

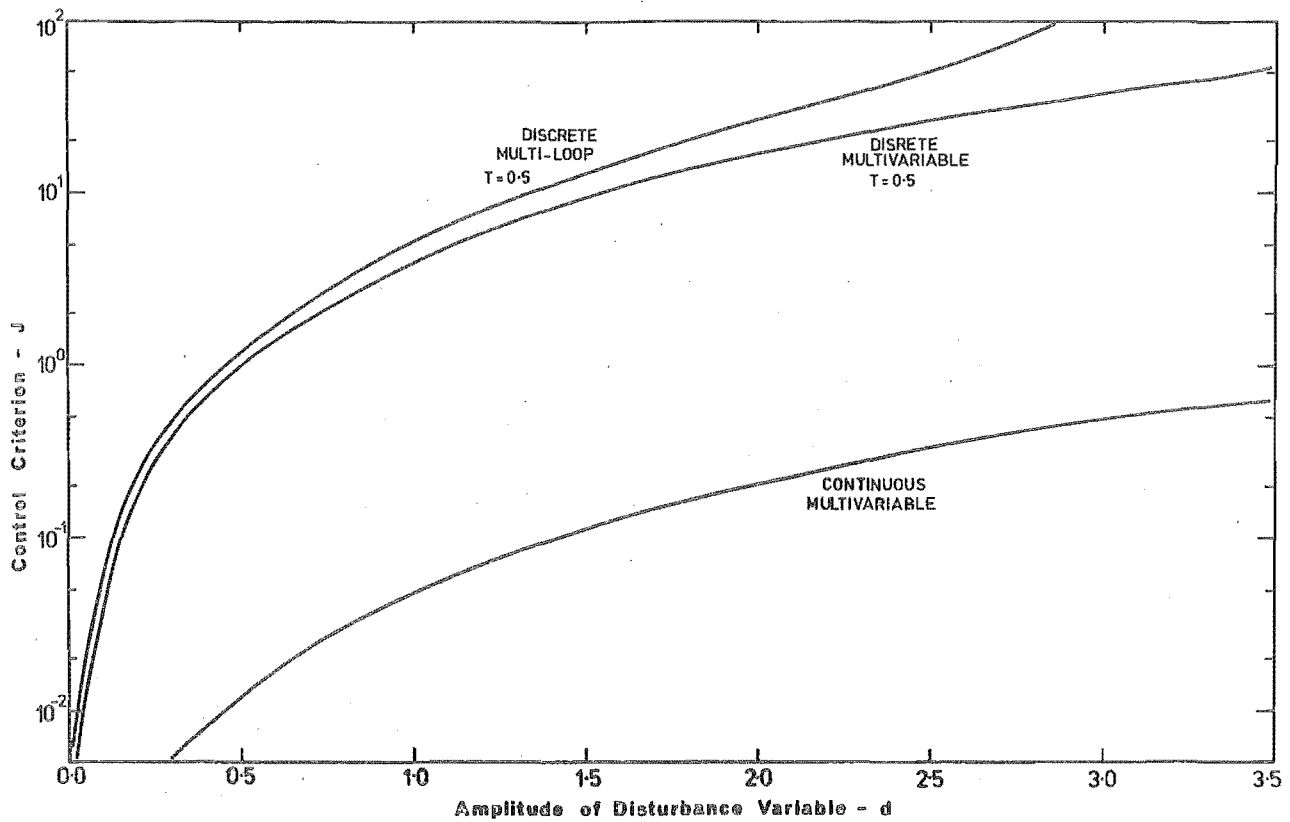


FIGURE 6.10 INCREASE IN CONTROL CRITERION WITH DISTURBANCE AMPLITUDE FOR REACTOR-2-2, SHOWING IMPRESSIVE PERFORMANCE OF CONTINUOUS MULTIVARIABLE CONTROLLER. COMPARISON WITH FIGURE 6.5 DEMONSTRATES THE CONSIDERABLE ADVANTAGE OF THE EXTRA CONTROL VARIABLE.

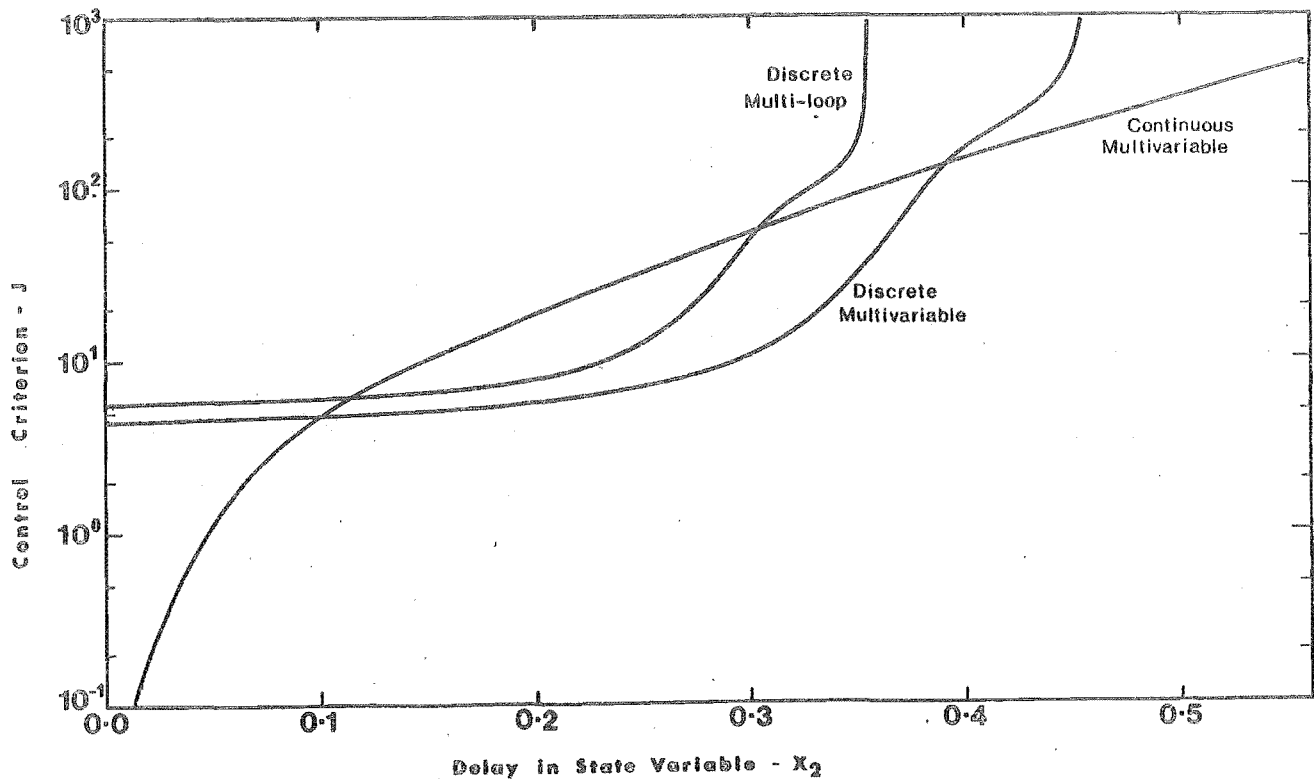


FIGURE 6.11 THE EFFECT OF STATE VARIABLE "DEAD-TIME" ON THE PERFORMANCE CRITERION FOR REACTOR-2-2 WITH RANDOM STEP DISTURBANCES. NOTE THE INITIAL SENSITIVITY BUT EVENTUAL STABILITY OF THE MULTIVARIABLE CONTINUOUS CONTROLLER.



a comparison of Figures 6.5 and 6.10. The continuous multivariable controller in particular showed excellent load rejection ability, with a criterion two orders of magnitude lower than the discrete controllers. The multi-loop discrete controller was not much inferior to the multivariable discrete controller as indicated also by their performance on the linear reactor model, although there was some divergence at higher noise levels.

Instead of delaying the control vector for REACTOR-2=2 the delay was inserted into the state variable  $x_2$ , corresponding to a delay involved in sampling and analysis of the product concentration. The effect of increasing state variable dead-time on the control criterion is shown in Figure 6.11. Again the performances of the two discrete controllers were very similar for small dead-times but the multivariable controller eventually showed a wide margin of stability over the multi-loop version.

The performance of the continuous controller in the face of dead-time was in complete contrast to its considerable ability in controlling random disturbances. The control criterion increased dramatically with the introduction of dead-time, and for state variable delays of between 0.1 and 0.35 its performance was poorer than the two discrete controllers. The discrete multi-loop controller became irretrievably unstable with a dead-time greater than 0.35 and the discrete multivariable at 0.45, whereas the continuous controller was still in control at a dead-time of 0.55.

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## CHAPTER VII

## EXOTHERMIC NON-LINEAR REACTOR

Exothermic continuous stirred-tank reactors are introduced and the well-established Aris and Amundson C.S.T.R. is used to compare various control techniques. An alternative C.S.T.R. is established with parameters chosen to provide instability and fast dynamics, and simplifying assumptions lead to a reduced order model. Multivariable and multi-loop controllers are developed and compared under conditions of sustained disturbances, and the effect of increasing discrete interval is again observed. A "Bang-Bang" controller is derived from a linear feedback law and is shown to stabilise the process but introduces high-frequency perturbations. Feedforward controllers produced a significant improvement in performance.

	Page
7.1 The Aris and Amundson C.S.T.R.	105
7.2 EXOREACTOR	108
7.3 Selection of Numerical Parameters	112
7.4 Performance of Controllers with Sustained Disturbances	114
7.5 Comparison of Multivariable and Multi-loop Controllers	117
7.6 Application of a "Bang-Bang" Controller	119
7.7 Feedforward Control of the Exothermic Reactor	125

## CHAPTER VII

## EXOTHERMIC NON-LINEAR REACTOR.

The analysis of continuous stirred-tank reactors has been an important area for chemical engineering teaching, research and discussion since the work of MacMillan and Weber (1935). Not only is the CSTR a very typical piece of chemical processing equipment, but its suitability for lumped parameter representation coupled with a virtually limitless capacity for non-linearities has made it a fruitful area for research into chemical kinetics, thermodynamics, stability and control. The work of Aris and Amundson established the exothermic CSTR with two states as a viable process model and they used it for the demonstration of process dynamics using phase-plane techniques (1958a), proportional control strategies (1958b), and the development of state-space representation and automatic computation techniques (1958c). Siebenthal and Aris (1964) went on to apply Pontryagin's Minimum Principle for the optimisation and control of the exothermic CSTR, and discussed the possibilities of control using a small digital computer to store the switching criteria. They also introduced the problems associated with control variable chattering in the vicinity of the steady-state and the establishment of a limit cycle.

The same basic CSTR was used by Lapidus and Luus (1967) to demonstrate optimal control techniques and by Sienfeld (1969) for optimal control in the face of process variable dead-time. Han (1970) demonstrated stability criteria for the model using two feedback control loops and Seborg (1971) investigated reactor stability in the face of sustained disturbances. Luus and Cormack (1972) discovered multiple solutions when applying variational techniques, and Hyun and Aris (1972) used the model to demonstrate the effects of control variable hysteresis.

The direct search for a constant feedback matrix by Luus (1974) used the Aris and Amundson CSTR also but the technique is specific to the initial conditions and process duration and therefore of limited value.

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## 7.1 THE ARIS AND AMUNDSON CSTR.

The Aris and Amundson reactor was developed by establishing mass and energy balances for a hypothetical CSTR in which an exothermic reaction was controlled by the flow of coolant through a cooling jacket.

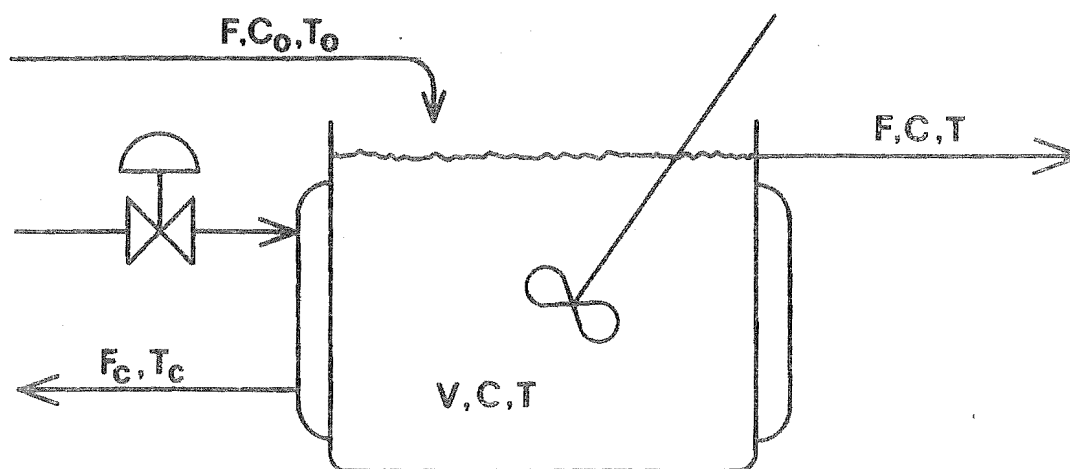


FIGURE 7.1. THE ARIS AND AMUNDSON C.S.T.R. WITH CONTROL OF THE COOLANT TEMPERATURE.

A component mass balance on the reactant provides

$$d(CV)/dt = F(C_0 - C) - VR(C, T) \quad \text{.....(7.1)}$$

and the energy balance gives

$$d(\rho C_p VT)/dt = \rho C_p F(T_0 - T) - VU^* - (-\Delta H)VR(C, T) \quad \text{.....(7.2)}$$

where  $(-\Delta H)$  is the heat of reaction and  $U^*$  is a heat removal rate per unit of reactor volume and is a function of  $T$ ,  $T_c$  and  $F_c$ . The  $R(C, T)$  term is a non-linear Arrhenius reaction rate function. The dynamics of the reactor shell and coolant are ignored and a simple linear relationship is developed between the rate of heat removed and coolant temperature. After manipulations to derive dimensionless variables, including dimensionless

time, the state equations become

$$\begin{aligned} dx_1/dt &= -(x_1 + 0.25) + (x_2 + 0.5)\exp(25x_1/(x_1 + 2)) \\ &= (1 + u)(x_1 + 0.25) \end{aligned} \quad \text{.....(7.3)}$$

and

$$dx_2/dt = (0.5 - x_2) - (x_2 + 0.5)\exp(25x_1/(x_1 + 2)) \quad \text{.....(7.4)}$$

Luus and Cormack (1972) used variational techniques to converge on the optimal non-linear controls to minimise a quadratic criterion

$$J = \int_0^{0.78} (x_1^2 + x_2^2 + 0.1u^2)dt \quad \text{.....(7.5)}$$

from an initial state of  $x_0^T = (0.09 \ 0.09)$ . An iterative procedure was established using

$$u(i+1) = u(i) - \epsilon \operatorname{sgn}(\partial H / \partial u) \quad \text{.....(7.6)}$$

where  $\epsilon$  is a relaxation factor and

$$\partial H / \partial u = -(x_1 + 0.25)\lambda_1 + 0.2u \quad \text{.....(7.7)}$$

The non-linear open-loop control law obtained by Luus and Cormack results in a minimum criterion of  $J = 1.331$  whereas feedback control based on a linearised model gave a criterion of  $J = 1.524$ . The advantages of the linear feedback control, presented in table 7.1, are considerable.

Luus (1974), employing a direct search technique on this occasion, produced a constant linear feedback control matrix  $K = \begin{bmatrix} 33.64 & 11.54 \end{bmatrix}$  and an optimal criterion of  $J^0 = 1.331$  but used over a minute of processor time on an IBM 370/165 computer indicating that, despite the simplicity of the search algorithm, each iteration is time consuming and the convergence is slow. The control law is once again specific to the initial conditions and the duration of the process. By comparison the solution of the Riccati equation for the steady-state feedback matrix based on a linear model took less than three seconds on a Burroughs B6718, and the result, although strictly optimal for the linearised system only, is entirely general and independent of time.

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TABLE 7.1

Comparison of Luus and Cormack non-linear open-loop controller and the steady-state linear feedback controller for the Aris and Amundson exothermic Reactor.

Luus and Cormack Non-linear controller	Steady-state Linear feedback controller
<p>Control law is open-loop and unsuitable for process stabilisation purposes. Sensitive to modelling errors.</p> <p>Control is specific to the initial conditions and to the duration of the operation.</p> <p>Maximum control effort is <math>u = 4.25</math></p> <p>Multiple solutions are possible resulting in sub-optimal control policies.</p> <p>Between 18 and &gt;100 iterations required depending on the initial value of <math>u(0)</math>.</p>	<p>Closed-loop controller ideal for process regulation and unaffected by disturbances or modelling errors.</p> <p>Steady-state control law is based on linear model and therefore not specific to time or initial conditions</p> <p>Maximum control effort is <math>u = 2.82</math></p> <p>Global optimality assured for the linearised closed-loop system.</p> <p>Seven iterations required to solve the Riccati equation to a precision of better than <math>10^{-6}</math>.</p>

## 7.2 EXOREACTOR.

The Aris and Amundson reactor model was restrictive in form and an alternative exothermic reactor model, named EXOREACTOR, was developed to include a second-order reaction and disturbance effects

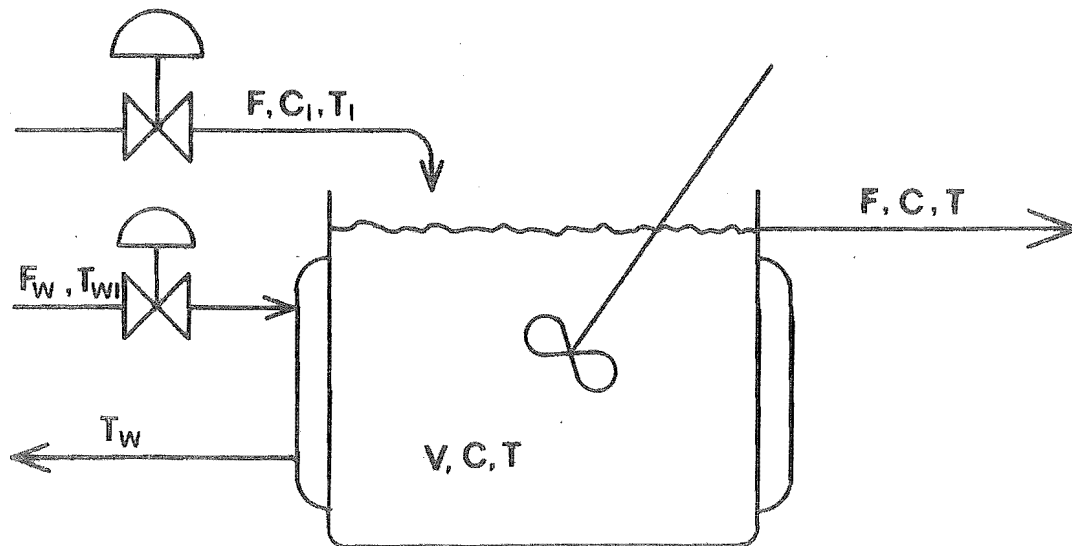


FIGURE 7.2. THE EXOTHERMIC REACTOR DEVELOPED TO INCLUDE TWO CONTROL VARIABLES AND TWO DISTURBANCE VARIABLES.

In order to establish the mass and energy balances for the reactor a number of basic assumptions were required.

(1) The reactor is an overflow type and the reactor volume is therefore constant despite changes in the reactant flow rate. A greater degree of control over product concentration and temperature would be possible with a variable volume reactor but at the expense of greater computational overheads.

(2) There is no phase change or density change associated with the reaction, and the density changes due to temperature may be regarded as insignificant.

(3) The coolant is well-mixed in the cooling jacket.

(4) The reaction rate is dependent on temperature and the rate constant is represented by Arrhenius' Law

$$k = k' \exp(-E/RT) \quad \text{.....(7.8)}$$

where  $k'$  is a frequency factor and  $E$  is the activation energy for the reaction. The Arrhenius Law temperature dependency has been found to provide excellent agreement with experimental data from a very wide range of chemical reactions and the Law may be derived from basic thermodynamic considerations, the principles of collision theory as well as transition-state theory, all of which are concisely presented by Levenspiel (1962).

(5) The thermal capacity of the reactor is negligible in comparison with that of the reactor contents or the cooling jacket contents. Coincident with the negligible thermal mass of the reactor wall, the resistance to heat transfer may be represented by a lumped overall heat transfer coefficient  $U$ .

A second-order reaction rate is proposed in order to accentuate the non-linearity of the reactor dynamics, i.e. under batch reaction conditions the reaction rate would be given by

$$r = dC/dt = -kC^2 \exp(-E/RT) \quad \text{.....(7.9)}$$

A component mass balance on the reactant gives

$$d(CV)/dt = F(C_I - C) - kVC^2 \exp(-E/RT)$$

which may be simplified to

$$dC/dt = F(C_I - C)/V - kC^2 \exp(-E/RT) \quad \text{.....(7.10)}$$

because of the constant reactor volume. An energy balance over the contents of the reactor provides

$$\begin{aligned} d(\rho C_p VT)/dt &= \rho C_p F(T_I - T) + (-\Delta H)VKC^2 \exp(-E/RT) \\ &\quad - UA(T - T_W) \end{aligned} \quad \text{.....(7.11)}$$

and an energy balance on the cooling jacket contents is

$$d(\rho_W C_{pW} V_W T_W)/dt = \rho_W C_{pW} F_W(T_{WI} - T_W) + UA(T - T_W) \quad \text{.....(7.12)}$$

The two most obvious variables to choose for control purposes are reactant flow rate,  $F$ , and coolant flow rate,  $F_W$ . Likely load variables are inlet reactant concentration and temperature  $C_I$  and  $T_I$ , and inlet coolant temperature,  $T_{WI}$ . An investigation of the reactor dynamics would indicate



however, that the volume of the cooling jacket is likely to be very small compared to the reactor and with a high coolant flow-rate the cooling jacket dynamics would be very fast relative to the reactor so that an important simplification is possible. The exit temperature of the coolant may be manipulated by an independent control loop and because of the fast response possible from a simple flow controller the disturbances due to fluctuations in coolant temperature should not affect the reactor significantly. Also, since the control variable becomes the set-point for the independent control loop, the model may conveniently be reduced to second order with consequent simplification of the control algorithm. The "stiff" system implied by equations 7.10, 7.11 and 7.12 might well have caused problems with the convergence rate of the matrix Riccati equations.

The non-linear state equations thus become

$$dC/dt = F(C_I - C)/V - kC^2 \exp(-E/RT) \quad \dots\dots(7.13)$$

and

$$dT/dt = F(T_I - T)/V - UA(T - T_W)/\rho VC_p + (-\Delta H)kC^2 \exp(-E/RT)/\rho C_p \quad \dots\dots(7.14)$$

The dimensionless deviation variables are chosen to be

$$\begin{aligned} x_1 &= (C - C_s)/C_s \\ x_2 &= (T - T_s)/T_s \\ u_1 &= (T_W - T_{Ws})/T_{Ws} \\ u_2 &= (F - F_s)/F_s \\ d_1 &= (C_I - C_{Is})/C_{Is} \\ d_2 &= (T_I - T_{Is})/T_{Is} \end{aligned}$$

and the state equations become

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{F_s}{V}(1 + u_1) \left[ \frac{C_{Is}}{C_s}(1 + d_1) - (1 + x_1) \right] \\ &\quad - \frac{F_s}{V} \left[ \frac{C_{Is} - C_s}{C_s} \right] (1 + x_1)^2 \exp(E_x/RT_s(1 + x_2)) \quad \dots\dots(7.15) \end{aligned}$$

and

$$\begin{aligned} \frac{dx_2}{dt} = & \frac{F_s}{V}(1 + u_1) \left[ \frac{T_{Is}}{T_s}(1 + d_2) - (1 + x_2) \right] \\ & - \frac{UA}{\rho VC_p} \left[ (1 + x_2) - \frac{T_{Ws}}{T_s}(1 + u_2) \right] \\ & + \frac{-\Delta H}{\rho C_p} \cdot \frac{F_s}{V} \left[ \frac{C_{Is} - C_s}{T_s} \right] (1 + x_1)^2 \exp(Ex_2/RT_s(1 + x_2)) \\ & \dots\dots\dots(7.16) \end{aligned}$$

For the purposes of developing the discrete-time models and the continuous and discrete control algorithms, the dynamic linear model may be found by establishing the Jacobean matrices and evaluating them at the steady-state operating conditions of the reactor. The elements of the continuous linear model are

$$\begin{aligned} a_{11} &= -F_s(2C_{Is} - C_s)/C_s V \\ a_{12} &= -F_s E(C_{Is} - C_s)/C_s VRT_s \\ a_{21} &= 2(-\Delta H)F_s(C_{Is} - C_s)/\rho C_p VT_s \\ a_{22} &= -F_s/V - UA/\rho VC_p + (-\Delta H)F_s E(C_{Is} - C_s)/\rho VC_p RT_s^2 \\ b_{11} &= F_s(C_{Is} - C_s)/C_s V \\ b_{12} &= 0.0 \\ b_{21} &= F_s(T_{Is} - T_s)/UT_s \\ b_{22} &= UAT_{Ws}/\rho VC_p T_s \\ c_{11} &= F_s C_{Is}/C_s V \\ c_{12} &= 0.0 \\ c_{21} &= 0.0 \\ c_{22} &= F_s T_{Is}/T_s V \end{aligned}$$

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## 7.3 SELECTION OF THE NUMERICAL PARAMETERS.

The selection of the process parameters to ensure instability and a reasonable amount of interaction was by no means straightforward. A reactor space-time of 10.0 was decided upon to slow the response, and the heat of reaction ( $-\Delta H$ ) and the heat transfer term  $UA$  were chosen to give a feasible coolant temperature. The important reactor parameters are listed in Table 7.2.

TABLE 7.2

Process parameters for EXOREACTOR to ensure instability, non-linearity and process variable interaction.

Reactor Volume	$V = 1.0$
Feed flow rate	$F_s = 0.1$
Mean residence time	$\tau = 10.0$
Feed Concentration	$C_{Is} = 80.0$
Feed Temperature	$T_{Is} = 50.0$
Product Concentration	$C_s = 50.0$
Product Temperature	$T_s = 70.0$
Heat of Reaction	$(-\Delta H) = 60.0$
Heat Transfer term	$UA = 5.0$
Arrhenius rate constant	$k = 2.09 \times 10^{-2}$
Coolant Temperature	$T_{Ws} = 34.4$

The dynamic linear process model based on these parameters is

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.220 & -0.171 \\ 5.143 & 2.247 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0.060 & 0.0 \\ -0.029 & 2.457 \end{bmatrix} \mathbf{u} + \begin{bmatrix} 0.160 & 0.0 \\ 0.0 & 0.071 \end{bmatrix} \mathbf{d}$$

and the open-loop eigenvalues are 0.214, 1.813 indicating a "stiff" system with both modes unstable. The instability is only local, however, and the reactor is globally stable and oscillatory under open-loop conditions exhibiting large-scale limit cycle behaviour. As Aris and Amundson (1958)

have shown, the potential instability is limited to the rate at which reactant is supplied to the reaction vessel.

Despite the reactor space-time of  $\tau = 10.0$ , the eigenvalues indicate fast dynamics and in preliminary experiments a discrete interval of  $T = 1.0$  did not allow adequate control in the face of random upsets.

The state and control variable penalty matrices were chosen to penalise most heavily the variables which were regarded as most critical. Thus concentration deviations were penalised more heavily than temperature deviations, and the variations in the reactant flow rate were more heavily penalised than fluctuations in the coolant flow. The penalty matrices were

$$Q = \begin{bmatrix} 10.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

and

$$R = \begin{bmatrix} 0.01 & 0.0 \\ 0.0 & 0.1 \end{bmatrix}$$

The discrete model was evaluated in 11 iterations to a precision of better than  $10^{-6}$  for a discrete interval of  $T = 0.2$  and the steady-state discrete feedback matrix was found after 15 iterations to be

$$K_{FBD}(0.2) = \begin{bmatrix} 23.318 & -0.524 \\ 1.487 & 2.145 \end{bmatrix}$$

A further 15 iterations were required to provide precision of the order of  $10^{-6}$ .

The size of the coefficients in the augmented matrix required the time interval for the Kalman-Englar technique to be reduced to 0.25 to ensure numerical stability, and the matrix exponential was evaluated after 14 iterations to a precision of  $10^{-8}$ . The steady-state Riccati equation was then solved to a precision of  $10^{-6}$  after 14 iterations resulting in a continuous feedback matrix.

$$K_{FBC} = \begin{bmatrix} 28.448 & -0.366 \\ 0.497 & 4.194 \end{bmatrix}$$

## 7.4 PERFORMANCE OF CONTROLLERS WITH SUSTAINED DISTURBANCES.

The controllers detailed above were compared, together with a multivariable discrete controller with time interval  $T = 0.5$ , under conditions of quite rapid fluctuations in the feed concentration and temperature. The concentration fluctuations took the form of random step changes above and below the mean with a frequency of 1.0. The feed temperature was varied sinusoidally with period around 2.0, and the amplitudes of both the concentration and temperature fluctuations were steadily increased throughout the control period.

The comparisons are given in table 7.3 and the process trajectories, showing also the disturbance variables which were identical for all runs, are shown in Figures 7.3, 7.4 and 7.5. The expected degradation in performance for the discrete controller with  $T = 0.5$  is graphically illustrated.

TABLE 7.3

Comparison of controller performance for EXOREACTOR under conditions of sustained disturbances in feed concentration and temperature.

Controller	Continuous	Discrete $T = 0.2$	Discrete $T = 0.5$
Control Criterion	0.1045	0.1324	3.86
State Variable			
range; $x(1)$	-0.032 to 0.022	-0.034 to 0.027	-0.089 to 0.045
$x(2)$	-0.032 to 0.019	-0.053 to 0.050	-0.211 to 0.42
Control Variable			
range; $u(1)$	-0.621 to 0.724	-0.614 to 0.760	-0.801 to 1.0
$u(2)$	-0.092 to 0.150	-0.129 to 0.137	-1.0 to 0.382

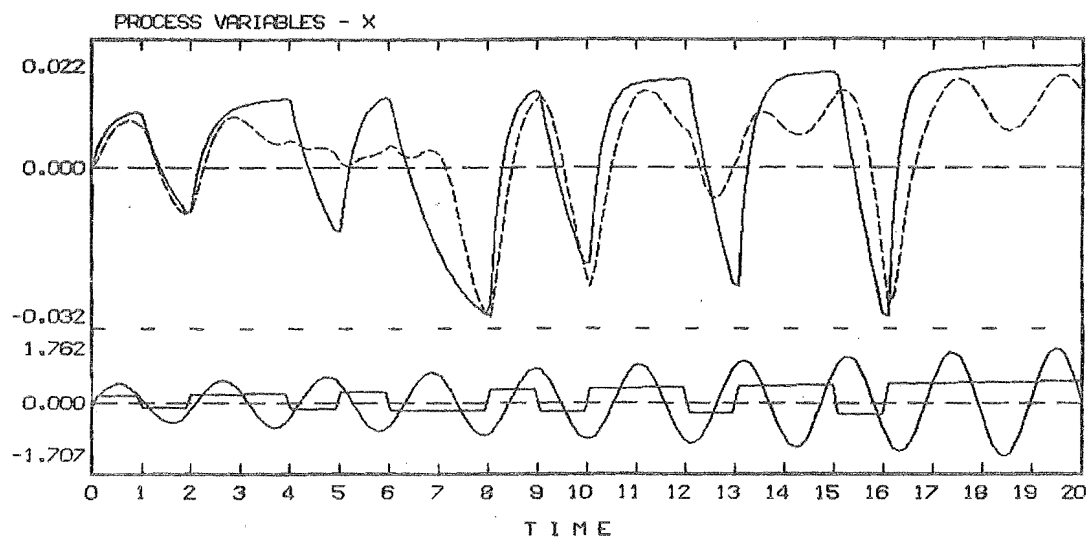


FIGURE 7.3. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR UNDER CONTINUOUS CONTROL SHOWING DISTURBANCE VARIABLES ALSO.

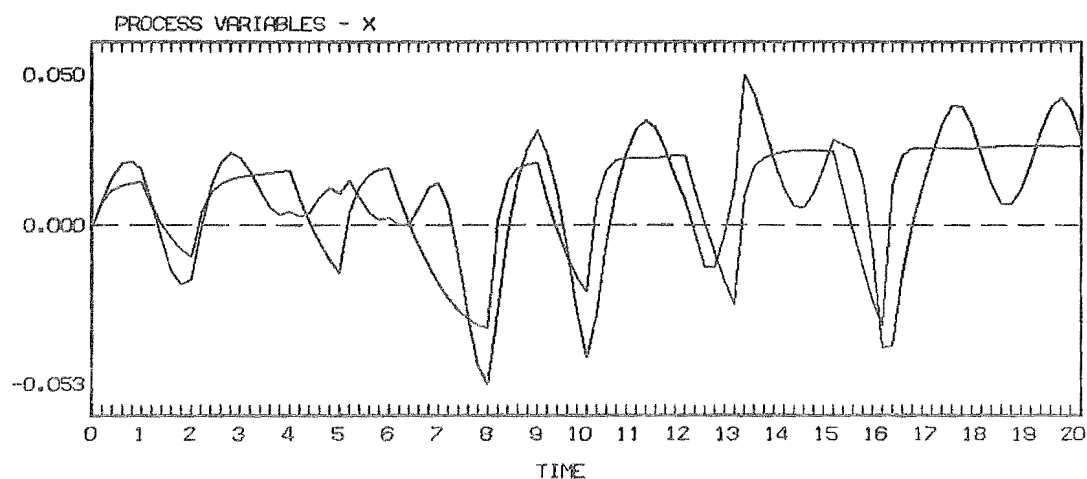


FIGURE 7.4. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR WITH A DISCRETE INTERVAL OF  $T = 0.2$ .

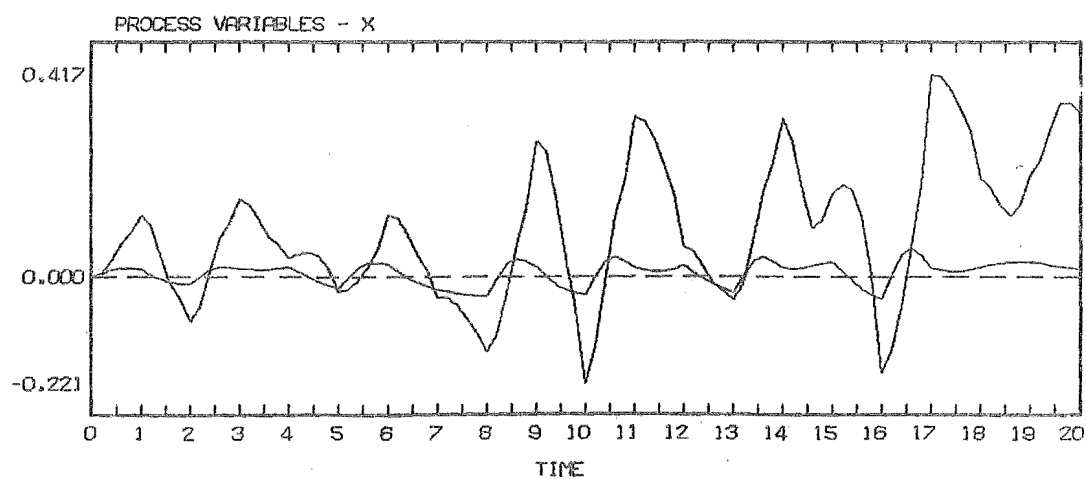


FIGURE 7.5. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR WITH A DISCRETE INTERVAL OF  $T = 0.5$  SHOWING GREATER PROCESS VARIABLE DEVIATIONS.

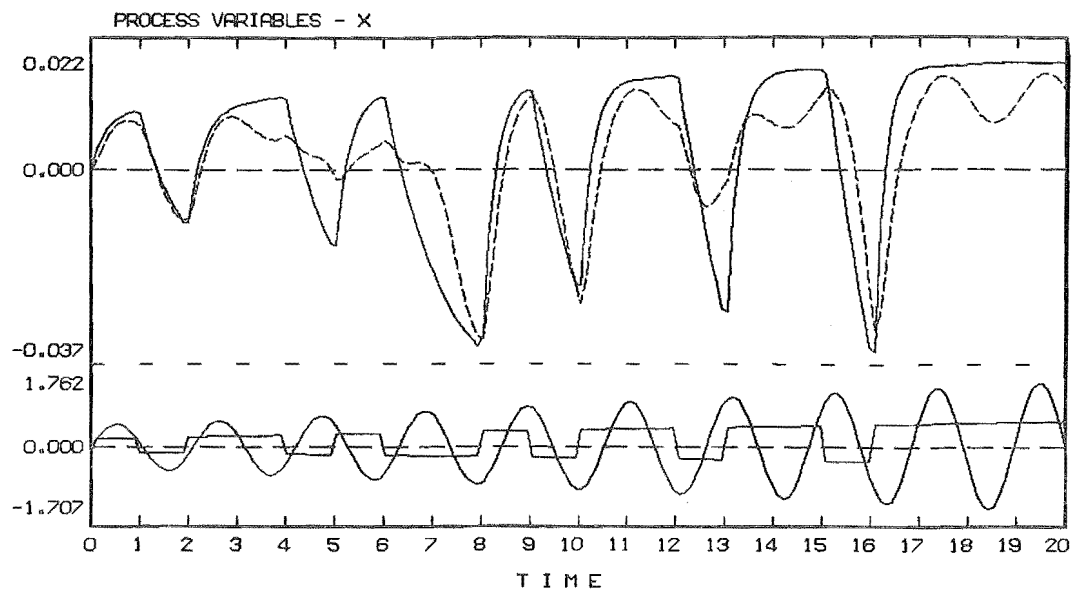


FIGURE 7.6. EXOREACTOR UNDER MULTI-LOOP CONTINUOUS CONTROL, WITH DISTURBANCE VARIABLES ALSO SHOWN.

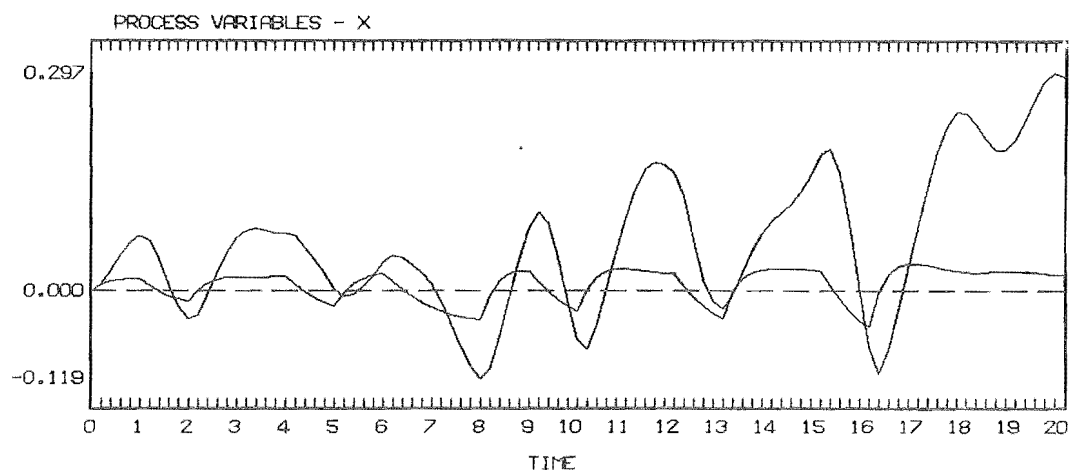


FIGURE 7.7. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR UNDER MULTI-LOOP CONTROL WITH A DISCRETE INTERVAL OF  $T = 0.2$ .

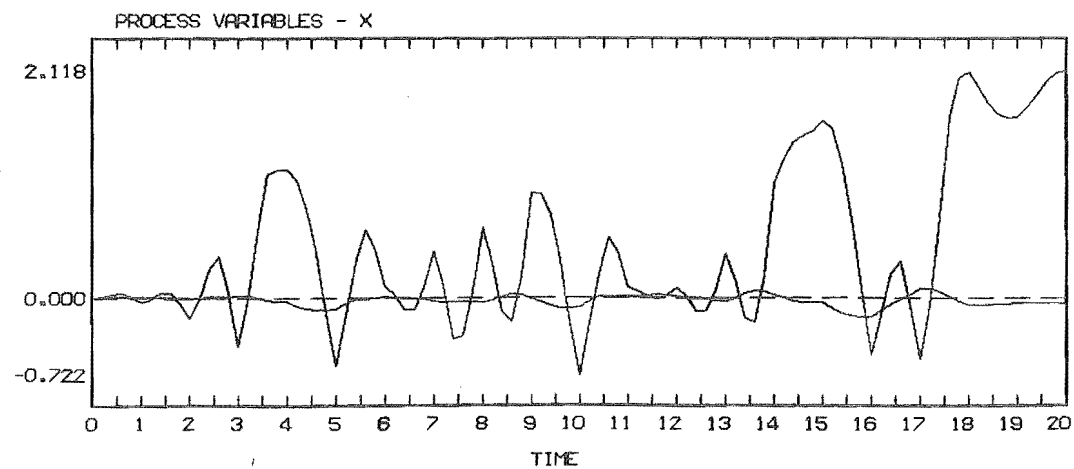


FIGURE 7.8. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR UNDER MULTI-LOOP CONTROL WITH A DISCRETE INTERVAL OF  $T = 0.5$ .

## 7.5 COMPARISON OF MULTIVARIABLE AND MULTI-LOOP CONTROLLERS.

Multi-Loop controllers, continuous and with a discrete interval of  $T = 0.5$ , based on the same control criteria were determined by iterative search techniques and compared with the multivariable controllers. The dominant  $k$  elements in both continuous and discrete feedback matrices, due to the heavier penalties on concentration and flow-rate fluctuations, resulted in multi-loop controllers very similar to the multivariable controllers. Comparisons using the linear reactor models showed the differences in performance to be very small for the continuous controllers and somewhat greater for the discrete controllers. The comparisons are shown in Table 7.4.

Process trajectories for the reactor under multi-loop control are shown in Figures 7.6, 7.7 and 7.8 and may be compared with the equivalent multivariable controlled reactors in Figures 7.3, 7.4 and 7.5. As expected for such similar feedback matrices there was little difference between the continuous multivariable and multi-loop controllers although the process and control variable excursions were slightly larger for the multi-loop controller, resulting in a greater process control criterion.

The differences between the discrete multivariable and multi-loop controllers were somewhat obscured by the multi-loop controlled processes for both  $T = 0.2$  and  $T = 0.5$  looking to become unstable at about time  $t = 17.0$ . Even before this occurred, however, the process and control variable excursions for both the discrete multi-loop controllers were considerably greater than those of the alternative multivariable controlled processes.



TABLE 7.4

Comparison of Multivariable and Multi-loop controllers based on the linear continuous and discrete models of EXOREACTOR.

Continuous Controllers	Multivariable	Multi-Loop
Feedback Matrix	$\begin{bmatrix} 28.448 & -0.366 \\ 0.497 & 4.194 \end{bmatrix}$	$\begin{bmatrix} 28.13 & 0.0 \\ 0.0 & 4.756 \end{bmatrix}$
Control Criterion	4.916	4.962
Norm of State at $t = 10.0$	$2.738 \times 10^{-5}$	$2.854 \times 10^{-5}$
Discrete Control- lers ( $T = 0.5$ )	Multivariable	Multi-loop
Feedback Matrix	$\begin{bmatrix} 17.424 & -0.897 \\ 1.646 & 1.294 \end{bmatrix}$	$\begin{bmatrix} 14.604 & 0.0 \\ 0.0 & 1.538 \end{bmatrix}$
Control Criterion	8.344	14.781
Norm of State at time $t = 20.0$	$1.95 \times 10^{-9}$	$4.54 \times 10^{-9}$

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## 7.6 APPLICATION OF A "BANG-BANG" CONTROLLER.

The standard approach for the development of a "Bang-Bang" control algorithm has been to reduce the penalty on the control variable to zero and impose control constraints instead. The Dynamic Programming approach will be demonstrated on a linear first-order process. The process may be described as

$$\dot{x} = ax + bu \quad \text{.....(7.17)}$$

with control criterion

$$J = \int_{t_0}^{t_f} x^2 dt \quad \text{.....(7.18)}$$

The control variable bounds are

$$-U \leq u \leq U \quad \text{.....(7.19)}$$

and the optimal criterion is given by

$$f(x,t) = \min_U \int_{t_0}^{t_f} x^2 dt \quad \text{.....(7.20)}$$

The Hamilton-Jacobi equation becomes

$$-\partial f/\partial t = \min_U \left[ x^2 + \partial f/\partial x (ax + bu) \right] \quad \text{.....(7.21)}$$

A simple inspection indicates that the bracketed terms may be minimised for

$$u = -U \operatorname{sgn}(\partial f/\partial x) \quad \text{.....(7.22)}$$

and the optimal control can be seen to switch between the extremes of  $U$  and  $-U$  according to the sign of  $\partial f/\partial x$ , which satisfies the partial differential equation

$$-\partial f/\partial t = x^2 - ax \frac{\partial f}{\partial x} - bU \left| \frac{\partial f}{\partial x} \right| \quad \text{.....(7.23)}$$

No simplifying assumptions as to the form of the optimal return function  $f(x,t)$  can be made for systems of this type and the analytical solution which exists in this instance results in an open-loop control law. For multivariable processes the situation is even more complicated, but a switching curve may be developed quite simply by the extremalisation of the Linear Quadratic approach. The controller gain is increased by increasing

the state penalty matrix and reducing the control penalty matrix, and the switching curve may be found by solving for

$$u_i = 0, i = 1, 2, \dots, m \text{ as a function of}$$

$x_j, j = 1, 2, \dots, n$ . The value taken by the control variables on either side of the switching curve are determined by an inspection of the original feedback gain matrix.

The technique was tested by driving EXOREACTOR to the origin from arbitrary initial conditions and this was accomplished as expected for a second-order process with two switches of the control variables. At the origin, however, the familiar chattering occurred as the control variables switched from one extreme to the other. The "Bang-Bang" controller was also tested under conditions of sustained disturbances and the trajectory is shown in Figure 7.9. The very fast fluctuations of the process variables would indicate that some of the control variable switchings have been missed due to the lack of discrimination of the plotting routines. The state variable excursions and the control criterion of  $J = 2.243$  may be compared to those of the multivariable controllers in Table 7.3.

A particular advantage of this method is its suitability for discrete control processes. Most of the time-optimal control algorithms require definite switching times which would seldom coincide with a control instant. With this technique, however, the control periods would have to be quite short or the control variable ranges suitably reduced to ensure that the process was not driven too far past the origin during a control interval.

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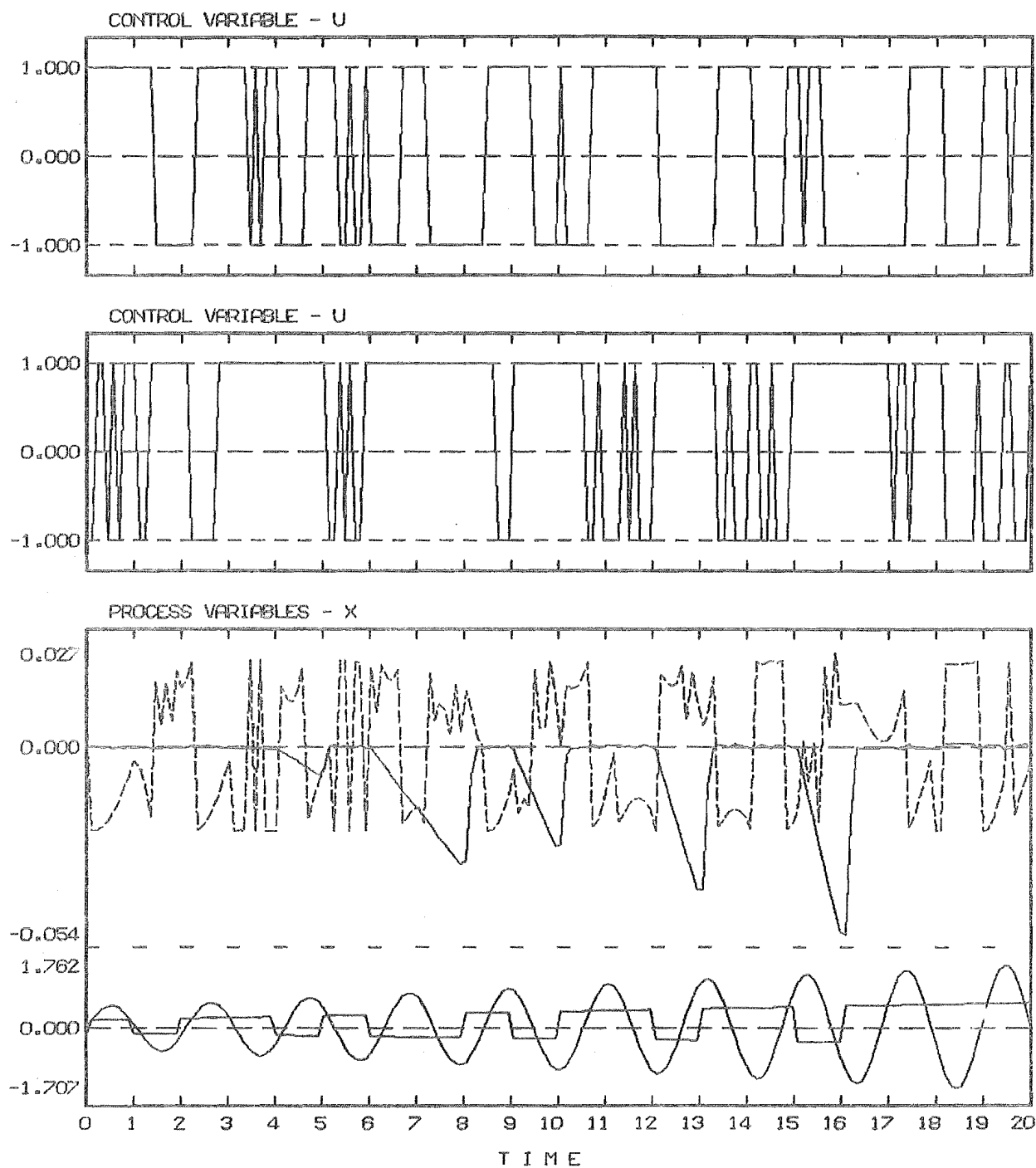


FIGURE 7.9. PROCESS VARIABLE TRAJECTORIES FOR EXOREACTOR UNDER "BANG-BANG" CONTROL SHOWING HIGH-FREQUENCY DISTURBANCES DUE TO THE RAPID CONTROL VARIABLE SWITCHING.

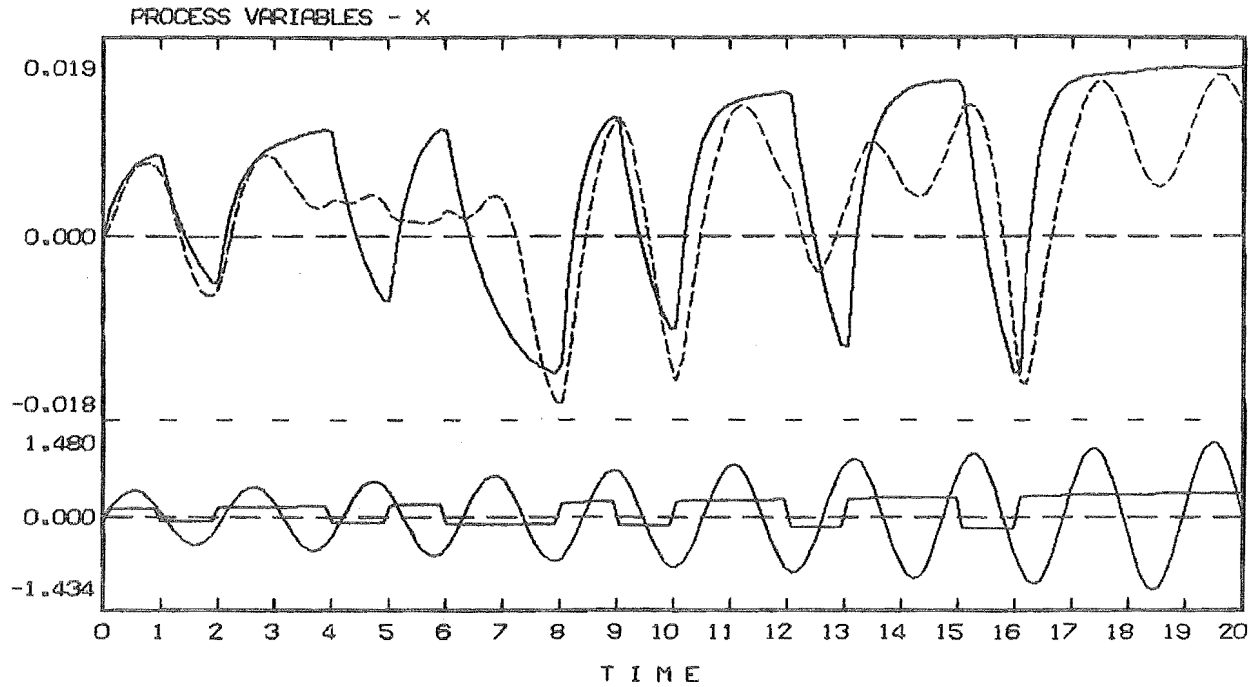


FIGURE 7.10. TRAJECTORIES FOR EXOREACTOR WITH CONTINUOUS FEEDBACK CONTROL ONLY SHOWING DISTURBANCE VARIABLES.

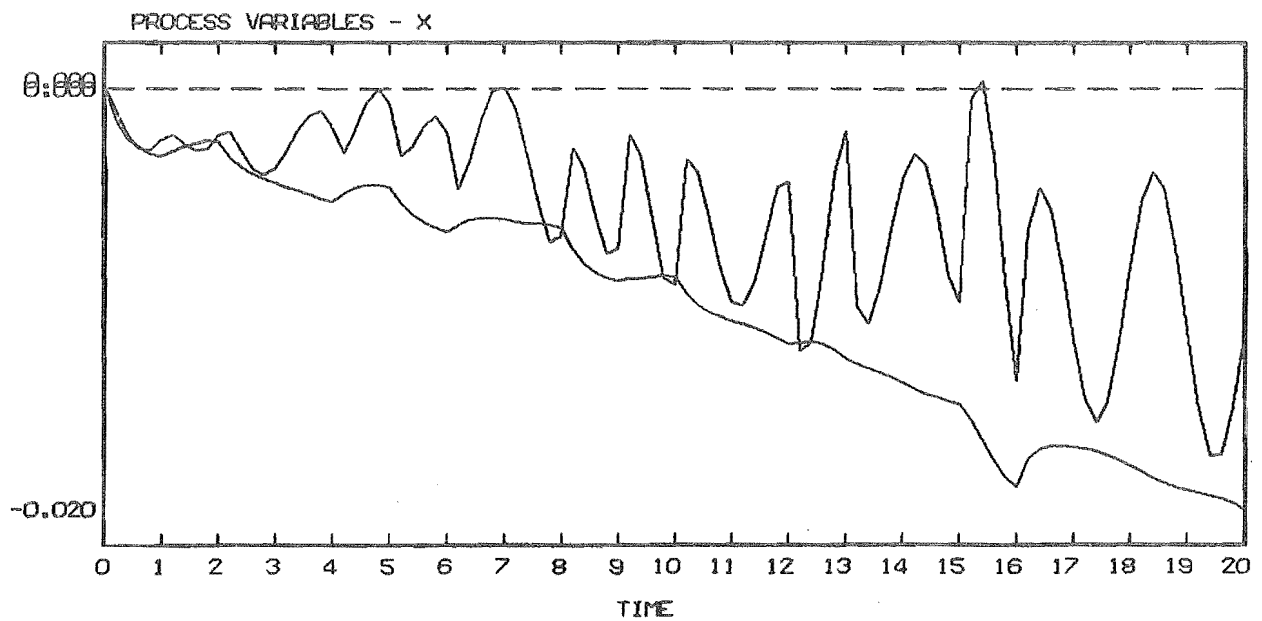


FIGURE 7.11. TRAJECTORIES FOR EXOREACTOR WITH CONTINUOUS FEEDFORWARD AND FEEDBACK CONTROL SHOWING STEADILY INCREASING OFFSET DUE TO THE FEEDFORWARD CONTROL BIAS.

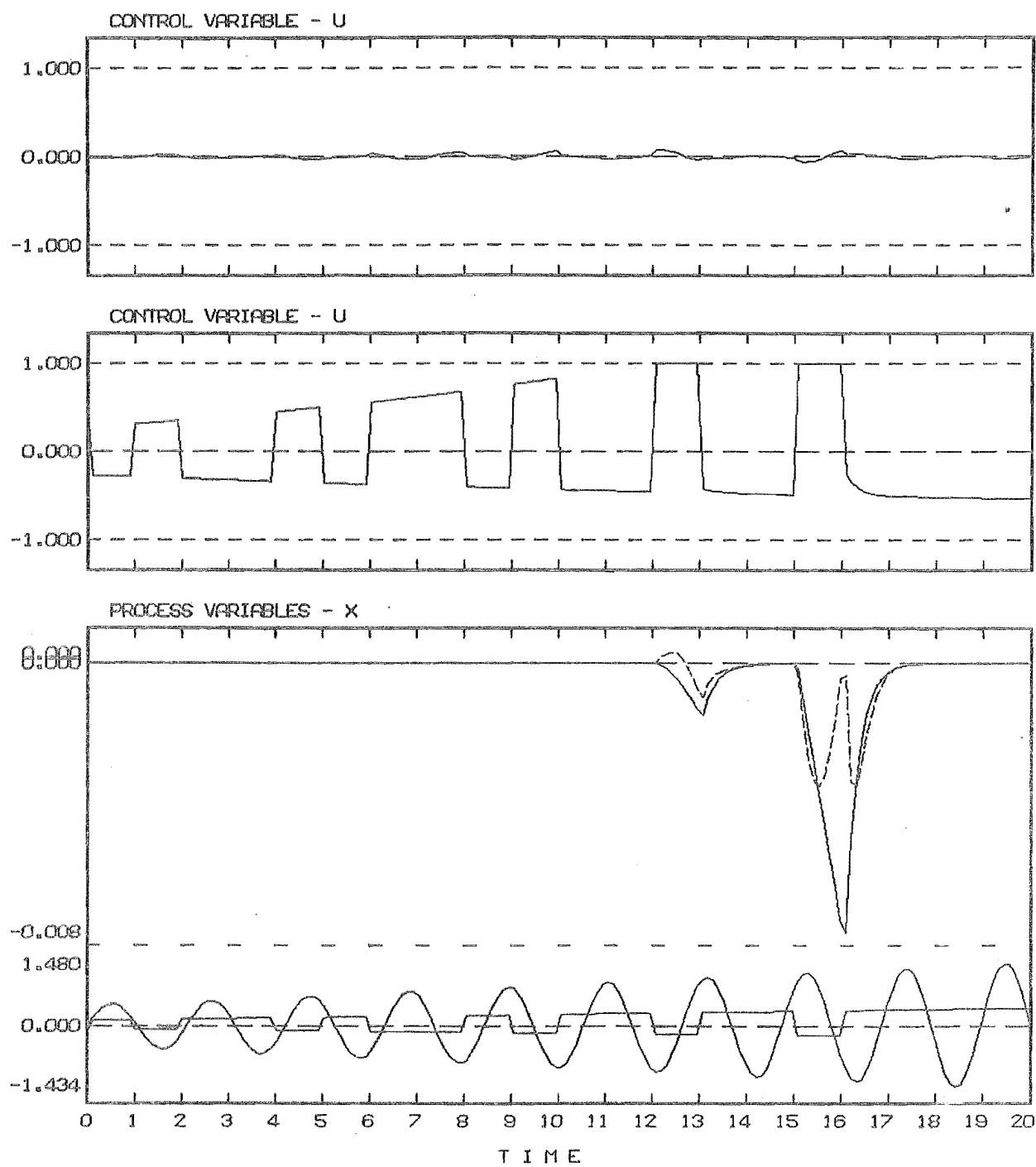


FIGURE 7.12. EXOREACTOR WITH NON-LINEAR FEEDFORWARD CONTROL SHOWING EXCELLENT LOAD VARIABLE COMPENSATION UNTIL THE CONTROL VARIABLE SATURATES.

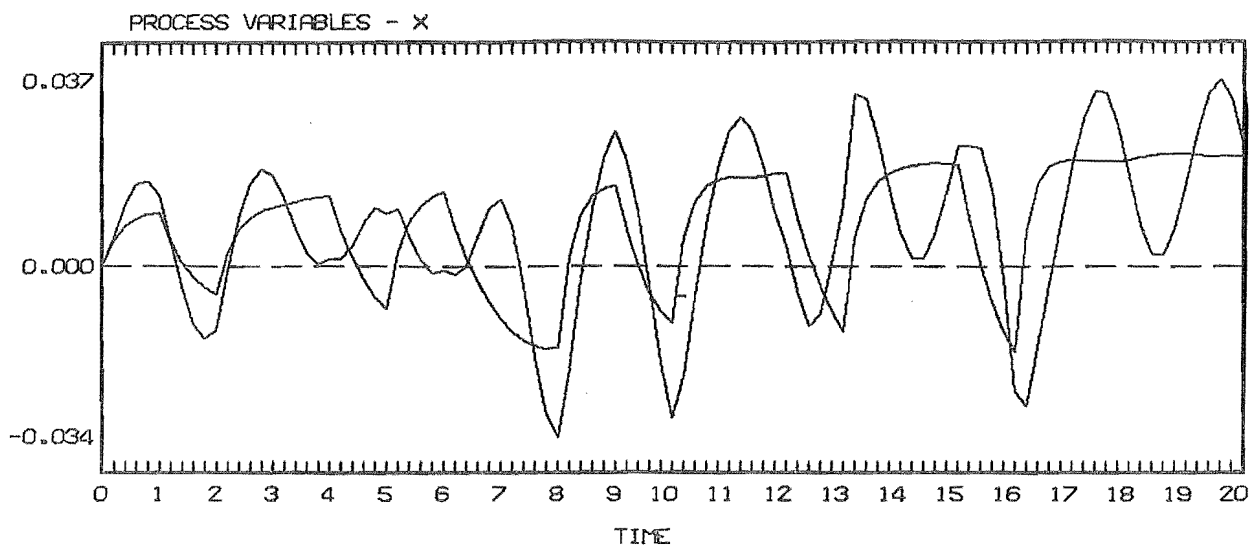


FIGURE 7.13. TRAJECTORIES FOR EXOREACTOR UNDER DISCRETE ( $T = 0.2$ ) FEEDBACK CONTROL ONLY.

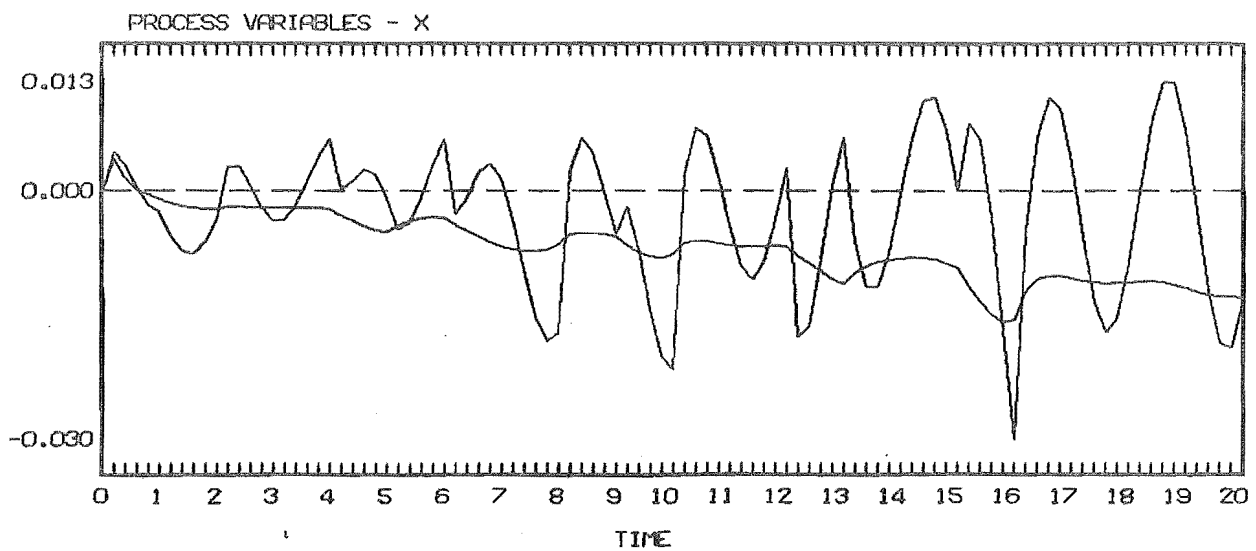


FIGURE 7.14. TRAJECTORIES FOR EXOREACTOR WITH DISCRETE FEEDFORWARD AND FEEDBACK CONTROL. NOTE THE REDUCED PROCESS VARIABLE DEVIATIONS BUT INCREASING OFFSETS.

## 7.7 FEEDFORWARD CONTROL OF THE EXOTHERMIC REACTOR.

Feedforward controllers, designed by the methods described in Chapter IV, were tested on EXOREACTOR using increasing random step changes of the feed concentration and sinusoidal variation of the feed temperature as disturbance variables. The linear feedforward controller, based on a linear model at the desired steady-state conditions, demonstrates clearly one of the problems associated with feedforward control. For disturbances in one direction the controller is over-compensating and for the other direction under-compensating, and the end result is a steadily increasing offset in the product concentration and temperature. This effect is demonstrated in figures 7.11 and 7.14 for the continuous and discrete controllers respectively. It is just this situation which would require integral states to be added to the process in order to drive the system back to the origin. Despite the tendency for the feedforward controlled process to diverge, however, the overall results are encouraging. As indicated in Table 7.5 the addition of feedforward control reduces the control criterion and reduces also the deviations of the process variables. The beneficial effect of feedforward control is less apparent with the continuous controlled process, but this is expected since, for a single-stage process the feedback controller is very effective in detecting the effects of disturbances and taking prompt remedial action. The advantages of feedforward control are more apparent with the discrete process and the advantages would be expected to increase with the discrete interval provided the frequency of the disturbances was lower than the sampling frequency. The most dramatic improvement in performance was achieved with the non-linear feedforward controller. The situation was unrealistic, perhaps, in that an exact process model was used to design the feedforward controller, however the effect of virtually zero process variable deviations (until control variable saturation) indicates the considerable potential of the technique.

Feedforward control would be of greater value in a multi-stage process, a distillation column for example, where the disturbances might have to work their way from the feed tray to the reflux drum before feedback control action could be taken; under these conditions the addition of feedforward control could be very beneficial.



TABLE 7.5

A comparison of the effects of linear and non-linear feedforward controllers using EXOREACTOR. The data was obtained from the trajectories shown in Figures 7.10 to 7.14.

Continuous Controllers	Feedback only	Linear Feedforward	Non-linear Feedforward
Control Criterion	$5.94 \times 10^{-2}$	$5.43 \times 10^{-2}$	$7.02 \times 10^{-4}$
State variable range; $x_1$	-0.0149 0.0185	-0.0203 0.00	-0.0083 0.00
$x_2$	-0.0183 0.0176	-0.0177 0.00043	-0.0039 0.00038
Discrete Controllers	Feedback only	Linear Feedforward	
Control Criterion	0.0739	0.0544	
Process variable ranges; $x_1$	-0.0151 0.0224	-0.0159 0.0037	
$x_2$	-0.0339 0.0374	-0.0298 0.0131	

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## CHAPTER VIII

## DISCUSSION AND CONCLUSIONS

The aims of the study are restated and a rationale for the development of the investigation is presented. The more significant results from the application of multivariable and multi-loop control laws to unstable linear processes and chemical reactors are presented, and the advantages of multivariable control techniques are discussed in greater detail.

Suggestions for future work include the continuing investigation of complex processing models by computer simulation until such time as the Modern Control Techniques are more readily accepted by the processing industries and more realistic comparisons are available.

	Page
8.1 The Aims of the Investigation	128
8.2 The Control of Simple Linear Processes	129
8.3 Non-linear Isothermal Reactors	130
8.4 Exothermic Non-linear Reactors	131
8.5 Comparisons between Multivariable and Multi-loop Control Systems	133
8.6 Suggestions for Future Work	134

## CHAPTER VIII

## DISCUSSION AND CONCLUSIONS

## 8.1 THE AIMS OF THE INVESTIGATION.

This investigation evolved in a somewhat unstructured fashion. The author had set out to develop improved control algorithms for chemical engineering processes and the objective was a design method for controllers which would be more robust and effective in the face of parameter variations, process noise and load changes, dead-time, and control and state variable constraints. It was soon apparent, however, that not only was this a well-tilled piece of ground but that the single-loop approach had very fundamental deficiencies when applied to the multivariable systems which comprise a modern integrated processing plant. Maintaining a distillation column reboiler temperature at a pre-determined set-point may be satisfactory under ideal conditions of no load changes, but it is quite unrealistic as a means of maintaining top or bottoms product quality in the face of changing feedstock flows or compositions.

The investigation then progressed toward the multivariable approach to control system synthesis; more particularly to the variational techniques and dynamic programming because of their ability to optimise a specific functional of the process states. Multivariable optimal control has important advantages in terms of the uniqueness and global optimality of the control laws, the asymptotic stability of the closed-loop system and the elimination of the configuration problem. Dynamic programming is of particular interest in view of the variety of techniques which may be employed. Discrete or continuous systems are treated with equal facility, non-linearities may be accommodated, and for the non-analytical approaches the existence of control variable bounds is a distinct advantage. The difficulties of the analytical approach, and the problems associated with the solution of the Hamilton - Jacobi equation are partly overcome by the use of digital computers for process control purposes. The terminal boundary conditions of the dynamic programming approach appear more

tractable than the split boundary conditions of the other variational techniques.

The investigation thus turned to the evaluation and implementation of multivariable optimal controllers based on the quadratic criterion which is generally favoured for regulatory control purposes.

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## 8.2 THE CONTROL OF SIMPLE LINEAR PROCESSES.

Second-Order linear processes were used to demonstrate the development of the control algorithms, and the number of iterations necessary to calculate the steady-state feedback gain matrices were recorded. Judicious choice of the integration intervals enabled the iterative techniques to converge rapidly and the algorithms were stable over a wide range of parameters. The effects of discrete interval were investigated and found to be of critical importance to the unstable process. This effect is not previously reported in this context but it would be a vital factor in the performance of systems where the discrete interval was large in relation to the process time constants.

The performance of single-loop controllers, developed to optimise the same criteria, were compared with the multivariable controllers and were markedly inferior. A pertinent point, also unreported elsewhere, was that the single-loop controllers required larger control variable excursions, but used them to much poorer effect in that the single-loop controlled processes incurred higher penalties, were returned to their set-points at a more sluggish rate and were generally more oscillatory. When control variable bounds were introduced the improvement in performance of the multivariable over the single-loop controllers was further demonstrated.

For the unstable process a steady-state feedback matrix was found in the vicinity of the optimal matrix which maintained control under tighter

control variable restrictions. The multivariable controller was able to maintain process stability under much tighter control bounds than was possible with the single-loop controller.

The use of the natural control bounds to limit the control variable was compared with the technique of increasing the control penalty in order to restrict the control action. The very small control penalty, with natural control bounds, resulted in a very fast return to the desired steady-state with a "bang-bang" style of control action. An additional feature of multivariable controllers, demonstrated at the same time, was their ability to accommodate very large feedback gain coefficients without the stability problems associated with large feedback gains on single-loop controllers.

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### 8.3 NON-LINEAR ISOTHERMAL REACTORS.

The techniques of process modelling and linearisation were applied to two non-linear isothermal reactors and the parameters for the autocatalytic reaction were chosen to ensure instability. The effect of a relatively large discrete interval on the control quality was demonstrated and zones of controllability for the discrete and continuous controllers also indicated the disadvantages of control with a large discrete interval. The performance increased dramatically when two control variables were employed, and the controllable region was significantly enlarged.

The ability of the controller to respond to process variable constraints was demonstrated, and particularly the facility with which a digital control computer is able to handle additional checks and manipulations.

Multivariable and single-loop controllers were compared using both the linear models and the non-linear process simulations, and again the

multivariable controllers showed significant advantages. The multivariable controllers were able to tolerate much larger disturbances before control variable saturation occurred and even when the control variables were regularly saturating the multivariable controllers maintained stability in the face of much more severe disturbances. The deterioration in the quality of control as the control interval was increased was also demonstrated under conditions of random noise disturbances. The reactor with two control variables showed a much improved load-rejection ability and an increase in stability over a wider range of disturbances.

The advantages of the multivariable controllers were demonstrated again when the reactors were subjected to random disturbances in the presence of control variable dead-time. In general the increasing dead-time caused a steady deterioration in the control quality. The rate of deterioration then increased dramatically until instability occurred. For the reactor with two control variables the continuous controller was found to be extremely sensitive to control variable dead-time, although ultimately it was able to maintain control at greater values of dead-time than either the multivariable discrete controller or the multi-loop continuous controller. This result was a little surprising for a controller with such exceptional load-rejection ability.

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#### 8.4 EXOTHERMIC NON-LINEAR REACTORS.

The well-used non-linear Aris and Amundson (1958) exothermic reactor was used to compare the linear feedback control method with the iterative approaches of Luus and Cormack (1972) and Luus (1974). The linear feedback approach, while slightly sub-optimal for the particular set of conditions, was shown to have considerable advantages in terms of computational effort, generality of solution and general applicability of the control law.

A less restrictive form of exothermic reactor was then developed to

allow higher order reaction kinetics, additional coolant dynamics and the introduction of disturbance variables. The reactor parameters were chosen to confer instability and "stiff" dynamics in order to adequately test the control algorithms. The state and control penalty matrices were chosen to penalise most heavily the deviations of product concentration and product flow rate, leading to a feedback matrix which was dominated by the  $K_{11}$  element. Increasing the discrete interval was again shown to have a very detrimental effect on the product quality and the reactor stability.

Multi-Loop controllers were developed, as for the isothermal reactors, but were very similar to the optimal controllers because of the dominant feedback elements. It was not unexpected, therefore, to find the performances of the multi-loop control laws only marginally poorer for the continuous controllers, and more noticeably so for the discrete controllers. Had the state and control penalty matrices been less heavily weighted against product concentration and flow-rate fluctuations the feedback matrices would not have had the single dominant element, and the difference in performance between the multivariable and multi-loop controllers would have been more dramatic.

A technique, previously unreported, for developing a switching curve for a "Bang-Bang" controller from an extremalised feedback control law was shown to control the exothermic reactor very adequately in the face of sustained and random process disturbances.

Linear feedforward controllers, both steady-state and dynamic, were developed for the exothermic reactor and their implementation brought significant improvements to the stability of the reactor subjected to random disturbances of feed concentration and temperature. A non-linear continuous feedforward compensation was also developed and the improvement in performance over the linear controllers was impressive.

The really significant advantages of feedforward control would not be expected from a single-stage process, however, and it would require a multi-stage process such as a distillation column or train of evaporators before the full advantages of feedforward techniques would be realised. The feedforward control algorithms were nevertheless shown to be straightforward

to implement and the advantages over feedback only control were substantial enough to warrant further investigation.

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## 8.5 COMPARISONS BETWEEN MULTIVARIABLE AND MULTI-LOOP CONTROL SYSTEMS.

In general the multivariable controllers offered significant advantages over the multi-loop controllers, but more particularly where the multivariable feedback matrices were not dominated by a single element as occurred with EXOREACTOR. For both the linear models and the non-linear simulations the multivariable controllers resulted in smaller control criteria, reduced state variable deviations and greater stability. Even under conditions of bounded control variables and state or control variable dead-time, for which the multivariable controllers were not specifically designed, the multivariable controllers demonstrated a considerable margin of stability over their single- or multi-loop counterparts. An important point, which has not previously been reported, is that the multivariable controllers have achieved this improved control with less control action than was used by the multi-loop controllers.

Another significant feature was that single- and multi-loop discrete controllers could not always be found for the unstable systems. Very extensive searches failed to locate feedback gains which could stabilise the process in situations where a multivariable controller with very modest feedback gains was able to bring it under control. The elimination of the configuration problem is a very major point in favour of the multivariable approach to control systems design. Even where a multi-loop control system is envisaged the calculation of a multivariable optimal feedback matrix is of considerable assistance in deciding which loops to close in order to approximate most closely to the multivariable control performance.

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## 8.6 SUGGESTIONS FOR FUTURE WORK.

The most obvious and pressing need is for comparative studies of both conventional single-loop and multivariable optimal controllers on a variety of processing plant. There are all too few papers reporting actual and practical applications of Modern Control Theories, but mention should be made of the work of Seborg, Fisher and co-authors of the University of Alberta, reported in a number of publications (1973, 1974a, 1974b, 1975). They have emphasised the practical application of fundamental theory to semi-scale integrated plant and the evidence is encouraging. The comparative studies of Bruun (1975) on a semi-scale double-effect evaporator showed that multivariable optimal control had many advantages over multi-loop control strategies.

The Universities are, however, restricted in the scale and type of the processing operations (hence the predominance of evaporator studies!) and are severely limited in the number and range of plant items which can conveniently be linked to provide a complex processing "unit". For this reason much of the reported research is performed on stable "unit operations" type of plant of low order. Until such time as the Modern Control Techniques are regularly applied in industrial processing situations, and their performance freely reported in the technical literature, much of the practical development work must be continued in University laboratories and will continue to be criticised for the reasons already stated.

Because of this the simulation of processing operations by digital and analogue computers may assist by providing the opportunity to investigate plant models of higher order and containing any desired forms of instability and non-linearity. In the controlled environment of a computer simulation the comparisons between various control techniques may be examined more rigorously and over a far greater range of processing conditions than could be tolerated on a real plant item. The ability to subject alternative controllers to identical sequences of plant disturbances, for example, results in a much more valid comparison of each controllers' abilities.

The ultimate test for any such new technology, however, is that it be able to take its place in the industrial environment and perform at least competitively against the conventional and established technologies. On this basis alone it is clear that there is a great deal of work still to be done.

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## APPENDIX I

## CONTINUOUS SYSTEMS MODELLING PROGRAM

## 8.1 DEVELOPMENT OF THE CONTINUOUS SYSTEM MODELLING PROGRAM - COSMO.

Early experiments in this series involving the digital simulation of chemical processes were made using an I.B.M. Package program called D.S.L. (for Digital Simulation Language) implemented on the I.B.M. 360/44 computer. D.S.L. was a sophisticated system, largely on account of the range of very skilfully employed integration routines, but also for the very extensive range of "functions" which could be employed. The system was so designed that writing a program to simulate a process was not unlike "wiring up" an analogue computer but very much easier in that no scaling of variables was necessary and all the flexibility and system intrinsics of FORTRAN-IV were available within the system. Every practical process operation was catered for by function subprograms, and the accuracy and stability of the integrating routines were guaranteed by error testing and step-halving. The overheads were high, however, and the system was overburdened by its own complexity to the extent that the change from the I.B.M. 360/44 to a Burroughs B6718 precipitated its demise.

To take its place a new "package" was created by the author involving a selection of sub programs which allowed process models to be simulated in a reasonably streamlined fashion. The basic unit is the integration function designed to solve the equation

$$x(t+\delta t) = \int_t^{t+\delta t} f(x,u,t)dt + x(t)$$

using the coding

$X(1) = \text{INTEGR}(1, XINIT, FOFX, LASTX),$  where FOFX may be

any function, explicit or implicit. A second-order Runge-Kutta integration was employed because of its simplicity, and no step-halving or step-doubling facility was used. A major advantage of the Second-order Runge-Kutta integration, using a small step-size, is that a discrete "event" was

unlikely to fall between integration intervals and if it did the time-lag involved in its implementation was minimal. A fourth-order Runge-Kutta routine would allow a much larger step-size with comparable accuracy and stability but the problem of discrete events occurring in the middle of the integration interval involves considerable programming complexity.

The integration was so arranged that the two iterations necessary to establish the gradients at beginning and end of the integration interval resulted in all the process variables being updated twice per step.

With the precision of a 48 bit word there was no need for double-precision and there was no significant round-off error accumulation even after 10 integrated steps.

A delay function was regrettably essential for the simulation of chemical plant and was simulated far better on a digital machine than with any Pade approximation. The function

$$x(t) = y(t - \tau) \quad \text{was coded}$$

$$X = \text{DELAY}(\text{TAU}, \text{XINIT}, Y)$$

The derivative of a variable

$$x = dy/dt \quad \text{was coded}$$

$$X = \text{DERIVT}(\text{XINIT}, Y) \quad \text{and was calculated using a}$$

linear or quadratic function where applicable.

A limiting function, coded

$$X = \text{LIMIT}(Y, \text{HILIMIT}, \text{LOLIMIT}), \quad \text{was useful for such}$$

variables as saturating valves and variables for which negative values cannot exist.

Functions for process disturbances also included steps, ramps and random noise.

The system is uncomplicated, easy to assemble and run and many times more accurate than a conventional analogue computer although a great deal slower. Anyone involved in the simulation of large systems by digital computer can expect to use a great deal of computer time. A sample program



listing is included and the "COSMO" subroutines are also listed on the following pages.

```

      FUNCTION INTEGR(K,IC,X,Y)
$ SET OWN
      COMMON DT,N,I
      REAL IC,INTEGR
      DIMENSION DY(6,2)
      IF(N.GT.0) GO TO 1
      INTEGR=IC
      GO TO 2
1     DY(K,I)=X
      INTEGR=Y+0.5*DT*(DY(K,1)+DY(K,I))
2     RETURN
      END

```

```

      FUNCTION DELAY(DELY,IC,XIN)
$ SET OWN
      COMMON DT,N,I
      REAL IC
      DIMENSION X(500)
      IF(N.GT.0) GO TO 1
      J=DELY/DT
      DO 2 M=1,J
2     X(M)=IC
      MARK=J+1
      DELAY=IC
      GO TO 4
1     IF(I.GT.1)GO TO 3
      K=MARK-J
      IF(K.LE.0)K=J+1+K
      DELAY=X(K)
      GO TO 4
3     X(MARK)=XIN
      DELAY=X(K)
      MARK=MARK+1
      IF(MARK.GT.(J+1))MARK=1
4     RETURN
      END

```

```

      FUNCTION LIMIT(X,HILIMT,LOLIMT)
C
C FUNCTION LIMIT CONSTRAINS "X" TO
C BETWEEN LOLIMIT AND HILIMIT
C
      REAL LIMIT,LOLIMT
      IF(X.GT.HILIMT)X=HILIMT
      IF(X.LT.LOLIMT)X=LOLIMT
      LIMIT=X
      RETURN
      END

```

```

      FUNCTION DERIVT(IC,XIN)
$ SET OWN
      COMMON DT,N,I
      REAL IC
      IF(N-2)1,2,3
1      K=1
      X3=XIN
      DERIVT=IC
      GO TO 4
2      K=2
      IF(I.EQ.1)GO TO 5
      X2=X3
      X3=XIN
5      DERIVT=(XIN-X2)/DT
      GO TO 4
3      K=3
      IF(I.EQ.1)GO TO 6
      X1=X2
      X2=X3
      X3=XIN
6      DERIVT=(3*XIN-4*X2+X1)/(2*DT)
4      RETURN
      END

```

```

      FUNCTION RAMP(X,TIME,START,SLOPE,FINISH)
C      FUNCTION GENERATES A LINEARLY INCREASING SIGNAL
C      FROM "START" TO "FINISH".
C
      IF(TIME.GT.START) GO TO 1
      RAMP=X
      GO TO 3
1      IF(TIME.GT.FINISH) GO TO 2
      Y=(TIME-START)*SLOPE
      RAMP=X+Y
      GO TO 3
2      RAMP=X+(FINISH-START)*SLOPE
3      CONTINUE
      RETURN
      END

```

```

      FUNCTION NOISE (X,AMPL,MEAN,NPERID)
C      "NOISE" GENERATES A RANDOMLY VARYING SIGNAL
C      UNIFORMLY DISTRIBUTED ABOUT "MEAN" WITH AMPLITUDE
C      "AMPL". THE VALUE OF NOISE IS CONSTANT OVER A
C      TIME-SPAN OF NPERIOD * DT
      REAL NOISE,MEAN
      COMMON DT,N,I
      IF(N.GE.1) GO TO 2
      Q=1.0
      K=0
      DO 1 J=1,100
1      Y=RANDOM(Q)
      Y=AMPL*(Y+MEAN-0.5)
      NOISE=X
      GO TO 4
2      K=K+1
      IF(K.LT.NPERID) GO TO 3
      K=0
      Y=AMPL*(RANDOM(Q)+MEAN-0.5)
3      NOISE=X+Y
4      CONTINUE
      RETURN
      END

```

```

C      SUBROUTINE CONTIN(N,M,PRI,FINT,KFBC,COST,ENORM,IND3,IND4)
C      CONTIN 2B-NOV-76
C      THE LINEAR PROCESS IS SIMULATED CONTINUOUSLY WITH AN
C      INTEGRATION STEP SIZE DT, PRINTOUT INTERVAL PRI, AND
C      FINAL TIME FINT.
C
      DIMENSION X(6),Y(6),X1(6),U(6),Z(6),ARG(6),G(6),D(6)
      COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
      1Q(6,6),R(6,6),XPLOT(6,200),UPLOT(6,200),DPL0T(6,200),
      2UMAX(6),UMIN(6),XD(6)
      COMMON DT,NN,I
      REAL KFBC(6,6)
      WRITE(6,302)N,M,PRI,FINT,IND3,IND4

302  FORMAT(44H) SUBROUTINE CONTIN(N,M,PRI,FINT,IND3,IND4)/
      110X,213,2F10.4,213/)
      PLTI=FINT/100.0
      IF(N.LT.3)GO TO 50
      WRITE(6,300)
300  FORMAT(47H) SYSTEM TOO LARGE FOR RICATI TO COPE.NO KFBC./)
      RETURN
      CONTINUE
      IF(IND3.LT.1)GO TO 14
      WRITE(6,203)
203  FORMAT(29H) CONTINUOUS FEEDBACK MATRIX/)
      DO 15 K=1,M
15   WRITE(6,202)(KFBC(K,J),J=1,N)
202  FORMAT(11H,10X,6F12.6)
      WRITE(6,204)
204  FORMAT(26H) CONTROL VARIABLE LIMITS/)
      DO 16 K=1,M
16   WRITE(6,205)K,UMIN(K),UMAX(K)
205  FORMAT(5H) U(,11,")",5X,2F10.4)
14   CONTINUE
      XO(N+1)=0.
      DO 1 K=1,N+1
1   X(K)=XO(K)
      WRITE(6,200)
200  FORMAT(21H) CONTINUOUS PROCESS//
      1" TIME COST STATE VARIABLES"/)
      TIME=0.
      COST=0.
      QUID=0.0
      PRT=0.
      PLTI=0.0
      PLTNUM=1
      NN=0
      WRITE(6,201)TIME,COST,(X(K),K=1,N)
2   DO 3 K=1,N+1
3   X1(K)=X(K)
      DO 9 I=1,2
      CALL MATVEC(KFBC,X,U,M,N)
      DO 4 K=1,M
      IF(U(K).LT.UMIN(K))U(K)=UMIN(K)
4   IF(U(K).GT.UMAX(K))U(K)=UMAX(K)
      IF(IND3.LT.3)GO TO 13
      IF(TIME.GT.0.2)GO TO 13
      WRITE(6,303)(U(K),K=1,M)
303  FORMAT(70X,6F10.4)
13   CONTINUE
      IF(I.LT.2)GO TO 19
      IF(IND4.LT.2)GO TO 19
      IF(TIME.LT.(PLTI-0.0001))GO TO 19
      DO 20 K=1,N
20   XPLOT(K,PLTNUM)=X(K)
      DO 21 K=1,M
21   UPLOT(K,PLTNUM)=U(K)
      PLTI=PLTI+PLTI
      PLTNUM=PLTNUM+1
19   CONTINUE
      CALL MATVEC(A,X,Z,N,N)
      CALL MATVEC(B,U,Y,N,M)
      DO 5 K=1,M
5   ARG(K)=Z(K)-Y(K)
      CALL MATVEC(G,Q,S,N,DT)
      ARG(N+1)=0.
      DO 6 K=1,N
6   ARG(N+1)=ARG(N+1)+X(K)*Z(K)
      CALL MATVEC(R,U,Y,M,M)
      DO 7 K=1,M
7   ARG(N+1)=ARG(N+1)+U(K)*Y(K)
      DO 8 K=1,N+1
8   X(K)=INTEGR(K,XO(K),ARG(K),X1(K))
9   CONTINUE
      COST=X(N+1)
      IF(NN.LE.0)GO TO 10
      DO 17 K=1,N
17   G(K)=X(K)
      CALL QUDRTC(G,Q,S,N,DT)
      QUID=QUID+S
      DO 18 K=1,M
18   D(K)=U(K)
      CALL QUDRTC(D,R,S,M,DT)
      QUID=QUID+S
      TIME=TIME+DT
      IF(TIME.LT.(PRT-1.0E-6))GO TO 11
10  WRITE(6,201)TIME,COST,(X(K),K=1,N),(U(K),K=1,M)
201  FORMAT(1H ,F6.2,1P10E12.4)
      PRT=PRT+PRI
11  NN=NN+1
      IF(TIME.LE.FINT)GO TO 2
      SOSQ=0.
      DO 12 K=1,N
12  SOSQ=X(K)*X(K)
      ENORM=SQRT(SOSQ)
      WRITE(6,206)COST,QUID
206  FORMAT(10H) COST = ,F20.10," QUID = "F20.10/)
      RETURN
      END

```

```

C      SUBROUTINE CONDIS(N,M,T,PRI,FIHT,KFBD,COST,XEND,IND3,IND4)
C      CONDIS 29-DEC-1976
C      A DISCRETE CONTROL ALGORITHM IS TESTED ON A
C      CONTINUOUS SYSTEM. IND3 CONTROLS PRINTOUT AND
C      IND4 STORES THE STATE AND CONTROL VARIABLES
C      FOR SUBSEQUENT PLOTTING.
C
C      DIMENSION X(6),U(6),X1(6),Y(6),V(6),ARG(6)
C      REAL KFBD(6,6)
C      COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
C      1Q(6,6),R(6,6),XPL0T(6,200),UPL0T(6,200),DPL0T(6,200),
C      2UMAX(6),UMIN(6),X0(6)
C      COMMON DT,NN,I
C      WRITE(6,200)N,M,T,PRI,FIHT,IND3,IND4
200  FORMAT(46H1 SUBROUTINE CONDIS(N,M,T,PRI,FIHT,IND3,IND4)/
C      15X,2I3,3F10.4,2I3/)
C
C      CALCULATE PLOTTING INTERVALS
C      IF(IND4.LT.2) GO TO 15
C      PLTI=FIHT/100.0
15  CONTINUE
C      CORRECT THE PRINTOUT INTERVAL TO FIT THE
C      DISCRETE TIME INTERVAL.
C
C      IF(PRI-T)1,2,3
1  PH=T/PRI
  NP=PH
  PRTI=T/NP
  GO TO 4
2  PRTI=T
  NP=1
  GO TO 4
3  PH=PRI/T
  NP=PH
  PRTI=T*NP
4  CONTINUE
  DO 5 K=1,N
5  X(K)=X0(K)
C      ZERO THE CONTROL VARIABLES
  DO 16 K=1,M
16  U(K)=0.0
  WRITE(6,201)
201  FORMAT(43H0 CONTINUOUS PROCESS WITH DISCRETE CONTROL/
C      15X,"TIME      COST      PRICE      STATE VARIABLES AND CONTROLS"/)
  CNT=0.
  TIME=0.
  COST=0.
  NPLT=1
  PLTT=0.0
  PRT=0.
  PRICE=0.
  NN=0
  KOUNT=0
  WRITE(6,202)TIME,KOUNT,COST,PRICE,(X(K),K=1,N)
202  FORMAT(1H ,F6.2,I3,3X,1P8E15.4)
  DO 7 K=1,N
7  X1(K)=X(K)
C      STORE PROCESS VARIABLES FOR PLOTTING
  IF(IND4.LT.2) GO TO 17
  IF(TIME.LT.(PLTT-0.0001)) GO TO 17
  PLTT=PLTT+PLTI
  DO 18 K=1,N
18  XPL0T(K,NPLT)=X(K)
  NPLT=NPLT+1
17  CONTINUE
  C1=COST
  IF(TIME.LT.(CNT-1.0E-4))GO TO 9
  CNT=CNT+T
  KOUNT=KOUNT+1
  CALL MATVEC(KFBD,X,U,N,N)
  DO 10 K=1,N
  IF(U(K).LT.UMIN(K))U(K)=UMIN(K)
10  IF(U(K).GT.UMAX(K))U(K)=UMAX(K)
C      STORE CONTROL VARIABLES FOR PLOTTING
  IF(IND4.LT.2) GO TO 19
  DO 20 K=1,M
20  UPL0T(K,KOUNT)=U(K)
19  CONTINUE
  CALL QUARTC(X,Q,S,N,T)
  CALL QUARTC(U,R,Z,N,T)
  PRICE=PRICE+S+Z
  CALL MATVEC(B,U,V,N,N)
  IF(IND3.LT.2)GO TO 9
  IF(KOUNT.GT.(2*NP))GO TO 9
  WRITE(6,202)TIME,KOUNT,COST,PRICE,(X(K),K=1,N),(U(K),K=1,M)
9  CONTINUE
  DO 8 I=1,2
  CALL MATVEC(A,X,Y,N,N)
  DO 11 K=1,N
  ARG(K)=Y(K)-V(K)
11  X(K)=INTEGR(K,X0(K),ARG(K),X1(K))
  CALL QUARTC(X,Q,S,N,1.0)
  CALL QUARTC(U,R,Z,N,1.0)
  ARGUE=S+Z
  COST=INTEGR(6,0.0,ARGUE,C1)
8  CONTINUE
  IF(NN.LF.0)GO TO 12
  TIME=TIME+DT
  IF(TIME.LT.(PRT-1.0E-4))GO TO 13
12  WRITE(6,202)TIME,KOUNT,COST,PRICE,(X(K),K=1,N),(U(K),K=1,M)
  PRT=PRT+PRTI
13  NN=NN+1
  IF(TIME.LF.FIHT)GO TO 6
  SOSQ=0.
  DO 14 K=1,N
14  SOSQ=SOSQ+X(K)*X(K)
  XEND=SQRT(SOSQ)
  RETURN

```

```

SUBROUTINE DISCRT(N,M,T,PRI,FINT,KFBD,COST,XEND,IND,IND4)
C DISCRT 29-NOV-76
C THE PROCESS IS SIMULATED IN DISCRETE TIME WITH
C PRINTOUT AT EACH INTERVAL T, OR SOME MULTIPLE OF
C THAT PRI, UP TO THE FINAL TIME FINT. IND IS
C USED TO CONTROL THE PRINTOUT. THE CONTROL
C CRITERION, COST, AND THE EUCLIDEAN NORM OF
C THE FINAL STATE, XEND, ARE CALCULATED.
C
  DIMENSION U(6),X(6),Y(6),V(6)
  COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
1Q(6,6),R(6,6),XPLQT(6,200),UPLQT(6,200),DPLQT(6,200),
2UMAX(6),UMIN(6),XO(6)
  REAL KFBD(6,6)
  WRITE(6,203)N,M,T,PRI,FINT,IND,IND4
203 FORMAT(46H1 SUBROUTINE DISCRT(N,M,T,PRI,FINT,IND1,IND4)/
110X,2I3,3F10.4,2I3/)
  IF(IND.LT.1)GO TO7
  WRITE(6,204)
204 FORMAT(27H0 DISCRETE FEEDBACK MATRIX/)
  DO 8 I=1,M
  8 WRITE(6,205)(KFBD(I,J),J=1,N)
205 FORMAT(1H0,10X,6F12.6)
  7 CONTINUE
  WRITE(6,200)
200 FORMAT(19H0 DISCRETE PROCESS//
1" TIME COST STATE VARIABLES"/)
  COST=0.
  PRT=0.
  TIME=0.
  K=0
  DO 1 I=1,N
  U(I)=0.0
  1 X(I)=XO(I)
  IF(PRI.LT.T)GO TO 9
  PN=PRI/T
  NP=PN
  PRTI=NP*T
  GO TO 10
  9 PRTI=T
  10 CONTINUE
  WRITE(6,201)TIME,COST,(X(I),I=1,N),(U(I),I=1,M)
201 FORMAT(1H ,F6.2,1P10E12.4)
  PRT=PRT+PRTI
  2 CONTINUE
  L=10+(N+1)*15
  CALL MATVEC(KFBD,X,U,M,N)
  DO 3 I=1,M
  IF(U(I).GT.UMAX(I))U(I)=UMAX(I)
  IF(U(I).LT.UMIN(I))U(I)=UMIN(I)
  IF(IND.LT.2)GO TO 3
  IF(TIME.GT.(10.0*T))GO TO 3
  IF(TIME.GT.(2.0*PRI))GO TO 3
  WRITE(6,201)TIME,COST,(X(I),I=1,N),(U(I),I=1,M)
  3 CONTINUE
  CALL QUDRTC(X,Q,S,N,T)
  CALL MATVEC(PHI,X,Y,N,N)
  CALL MATVEC(DEL,U,V,N,M)
  CALL QUDRTC(U,R,W,N,T)
  DO 4 I=1,N
  4 X(I)=Y(I)-V(I)
  TIME=TIME+T
  COST=COST+S+W
  IF(TIME.LT.PRT)GO TO 2
  IF(IND.LT.1)GO TO 5
  WRITE(6,201)TIME,COST,(X(I),I=1,N),(U(I),I=1,M)
  5 CONTINUE
  PRT=PRT+PRTI
  IF(TIME.LT.FINT)GO TO 2
  WRITE(6,201)TIME,COST,(X(I),I=1,N),(U(I),I=1,M)
  SOSQ=0.
  DO 6 I=1,N
  6 SOSQ=SOSQ+X(I)*X(I)
  XEND=SQRT(SOSQ)
  RETURN
  END

```

## APPENDIX II

## SYSTEM SUBROUTINES FOR MULTIVARIABLE OPTIMAL CONTROL.

A sample main-line program (EXOREACTOR) is presented together with the system subroutines which would be required to calculate the continuous and discrete linear process models and the continuous and discrete feedback and/or feedforward control matrices.

```

SUBROUTINE PHIDEL(N,M,K,T,IND)
C PHIDEL (WARD & STRUM) CALCULATES THE PHI,DEL & PSI MATRICES
C USING A SINGLE MATRIX SERIES EVALUATION
C
COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
1Q(6,6),R(6,6),XPLOT(6,200),UPLOT(6,200),DPLLOT(6,200),
2UMAX(6),UMIN(6),XO(6)
C DIMENSION PHITRM(6,6),DELTRM(6,6),PSITRM(6,6),G(6,6),AT(6,6)
C
C CALCULATE MATRIX NORM TO ENSURE CONVERGENCE
C
18 SOSQ=0.
DO 16 I=1,N
DO 16 J=1,N
16 SOSQ=SOSQ+A(I,J)*A(I,J)
ANORM=SQRT(SOSQ)
IF(ANORM*T.LT.20.0)GO TO 17
T=T/2.0
GO TO 18
17 CONTINUE
C
C INITIALISE MATRIX SERIES
C
DO 1 I=1,N
DO 1 J=1,N
IF(I.EQ.J)GO TO 2
PHI(I,J)=0.
PHITRM(I,J)=0.
GO TO 1
2 PHI(I,J)=1.0
PHITRM(I,J)=1.0
1 CONTINUE
DO 3 I=1,M
DO 3 J=1,M
DEL(I,J)=B(I,J)*T
3 DELTRM(I,J)=DEL(I,J)
IF(K.LE.0)GO TO 24
DO 20 I=1,N
DO 20 J=1,K
PSI(I,J)=C(I,J)*T
20 PSITRM(I,J)=PSI(I,J)
24 CONTINUE
DO 4 I=1,N
DO 4 J=1,N
4 AT(I,J)=A(I,J)*T
L=1
C
C SUM THE SERIES UNTIL CRITERION IS MET
C
10 CONTINUE
XL=L
RPL=1.0/XL
RPLP1=1.0/(XL+1.0)
CALL MULTIP(PHITRM,AT,G,N,N,N)
PABS=0.
DO 5 I=1,N
DO 5 J=1,N
PHITRM(I,J)=G(I,J)*RPL
G(I,J)=PHI(I,J)
5 IF(ABS(PHITRM(I,J)).GT.PABS)PABS=ABS(PHITRM(I,J))
CALL ADDMAT(G,PHITRM,PHI,N,N)
CALL MULTIP(AT,DELTRM,G,N,N,M)

```

```

      BABS=0.
      DO 6 I=1,N
      DO 6 J=1,M
      DELTRM(I,J)=G(I,J)*RPLP1
      IF(ABS(DELTRM(I,J)).GT.BABS)BABS=ABS(DELTRM(I,J))
6    G(I,J)=DEL(I,J)
      CALL ADDMAT(G,DELTRM,DEL,N,M)
      IF(K.LE.0)GO TO 25
      CALL MULTIP(AT,PSITRM,G,N,N,K)
      CABS=0.
      DO 21 I=1,N
      DO 21 J=1,K
      PSITRM(I,J)=G(I,J)*RPLP1
      IF(ABS(PSITRM(I,J)).GT.CABS)CABS=ABS(PSITRM(I,J))
21  G(I,J)=PSI(I,J)
      CALL ADDMAT(G,PSITRM,PSI,N,K)
25  CONTINUE

C
C    WRITE OUT CURRENT PHI,DEL & PSI MATRICES
C
      IF(IND.LT.2)GO TO 15
      WRITE(6,200)L,PABS,BABS,CABS
C 200 FORMAT(8H0 AFTER,I3," ITERATIONS MAXIMUM TERMS ARE",
C        13F20.10)
      DO 7 I=1,N
      C 7 WRITE(6,201)(PHI(I,J),J=1,N)
C 201 FORMAT(6H0 PHI,6F20.10)
      DO 8 I=1,N
      C 8 WRITE(6,202)(DEL(I,J),J=1,M)
C 202 FORMAT(6H0 DEL,6F20.10)
      DO 22 I=1,N
      C 22 WRITE(6,207)(PSI(I,J),J=1,K)
C 207 FORMAT(6H0 PSI,6F20.10)
      15 L=L+1
      C IF(PABS.GT.1.0E-10)GO TO 9
      IF(PABS.GT.1.0E-6)GO TO 9
      C IF(BABS.GT.1.0E-10)GO TO 9
      IF(BABS.GT.1.0E-5)GO TO 9
      GO TO 12
      9 IF(L.LT.100)GO TO 10
      WRITE(6,203)

      203 FORMAT(17H0 NO CONVERGENCE/)
      12 CONTINUE
C 12 WRITE(6,204)L,T
C 204 FORMAT(35H0 FINAL PHI AND DEL MATRICES AFTER,I5,
C        1" ITERATIONS"/20X," TRANSITION MATRIX - PHI(",F7.4,"")"/)
C      DO 13 I=1,N
C 13 WRITE(6,205)(PHI(I,J),J=1,N)
C 205 FORMAT(1H0,20X,6F20.10)
C      WRITE(6,206)T
C 206 FORMAT(1H0,20X," DRIVING MATRIX - DEL(",F7.4,"")"/)
C      DO 14 I=1,N
C 14 WRITE(6,205)(DEL(I,J),J=1,M)
C      WRITE(6,208)T
C 208 FORMAT(1H0,20X," DISTURBANCE MATRIX - PSI(",F7.4,"")"/)
C      DO 23 I=1,N
C 23 WRITE(6,205)(PSI(I,J),J=1,K)
      RETURN
      END

```

```

SUBROUTINE CONTR1(N,M,PHI,DEL,D,E,FBM,DT,IND)
C SUBROUTINE CALCULATES THE FEEDBACK GAIN MATRIX BY
C ITERATIVE SOLUTION OF THE DISCRETE RICATTI EQUATION
C AND APPLIES IT TO THE LINEAR SYSTEM UNDER STUDY
DOUBLE PRECISION DET
DIMENSION PHI(6,6),DEL(6,6),Q(6,6),R(6,6),S(6,6),T(6,6),FBM(6,6)
1 U(6,6),V(6,6),G(6,6),A(6,6),H(6,6),X(6),Y(6),C(6,6),B(6,6),
2 F(6,6),P(6,6),U1(6),D(6,6),E(6,6)
WRITE(6,200)
200 FORMAT(44H1 EVALUATION OF FEEDBACK COEFFICIENT MATRIX//)
WRITE(6,201)DT
201 FORMAT(27H0 TRANSITION MATRIX - PHI(,F6.4,")"/)
DO 1 I=1,N
1 WRITE(6,202)(PHI(I,J),J=1,N)
202 FORMAT(/10X,6F20.8/)
WRITE(6,203)DT
203 FORMAT(24H0 DRIVING MATRIX - DEL(,F6.4,")"/)
DO 2 I=1,N
2 WRITE(6,202)(DEL(I,J),J=1,M)
WRITE(6,204)
204 FORMAT(27H0 PENALTY MATRICES - G,A,H/)
DO 27 I=1,N
DO 27 J=1,N
G(I,J)=D(I,J)
27 A(I,J)=D(I,J)
DO 3 I=1,N
3 WRITE(6,205)(G(I,J),J=1,N)
205 FORMAT(/10X,6F10.4)
DO 4 I=1,N
4 WRITE(6,205)(A(I,J),J=1,N)
DO 5 I=1,M
DO 28 J=1,M
28 H(I,J)=E(I,J)
5 WRITE(6,205)(H(I,J),J=1,M)
K=0
DO 6 I=1,M
DO 6 J=1,N
FBM(I,J)=0.
6 CONTINUE
SUM=0.
DO 19 I=1,M
19 WRITE(6,206)K,SUM,(FBM(I,J),J=1,N)
DO 20 I=1,N
20 WRITE(6,207)(G(I,J),J=1,N)
K=K+1
21 CONTINUE
CALL MULTIP(G,DEL,Q,N,N,M)
CALL TRANSP(DEL,U,N,M)
CALL MULTIP(U,Q,R,M,N,M)
CALL ADDMAT(H,R,S,M,M)
CALL INVERT(6,M,S,T,DET)
CALL MULTIP(Q,T,C,N,M,M)
CALL TRANSP(C,V,N,M)
CALL MULTIP(V,PHI,B,M,N,N)
SUM=0.
DO 7 I=1,M
DO 8 J=1,N
DIFF=ABS(FBM(I,J)-B(I,J))
FBM(I,J)=B(I,J)
8 SUM=SUM+DIFF
IF(IND.LT.1) GO TO 7
WRITE(6,206)K,SUM,(FBM(I,J),J=1,N)
206 FORMAT(10X,13,3X,F12.8,5X,6F12.6)
7 CONTINUE
IF(SUM.LT.1.0E-6)GO TO 9
CALL MULTIP(U,G,Q,M,N,N)
CALL MULTIP(C,Q,S,N,M,N)
CALL SUBMAT(G,S,P,N,N)
CALL TRANSP(PHI,S,N,N)
CALL MULTIP(S,P,T,N,N,N)
CALL MULTIP(T,PHI,Q,N,N,N)
CALL ADDMAT(Q,A,G,N,N)

```



```

      IF(IND.LT.2) GO TO 25
      DO 10 I=1,N
12  WRITE(6,207)(G(I,J),J=1,N)
207 FORMAT(/60X,1P6E10.3)
25  CONTINUE
      K=K+1
      IF(K.GT.100) IND=0
      IF(K.LT.500) GO TO 21
      WRITE(6,213)
213  FORMAT(24H0 NO DAMN CONVERGENCE]])
      DO 26 I=1,M
26  WRITE(6,206)K,SUM,(FBM(I,J),J=1,N)
      9 CONTINUE
C
C  WRITE OUT FEEDBACK GAIN MATRIX
C
      WRITE(6,400)
400  FORMAT(27H DISCRETE FEEDBACK MATRIX/)
      DO 24 I=1,M
24  WRITE(6,214)K,(FBM(I,J),J=1,N)
214  FORMAT(1H0,15,5X,6F18.10)
C
C  CALCULATE CLOSED LOOP EIGENVALUES
C
      CALL MULTIP(DEL,FBM,B,N,M,N)
      CALL SUBMAT(PHI,B,A,N,N)
      CALL EIGVEC(6,N,A,X,Y,B)
      WRITE(6,215)
215  FORMAT(26H0 CLOSED LOOP EIGENVALUES/
1  " REAL IMAGINARY STABILITY "/)
      DO 23 I=1,N
      TEST=SQRT(X(I)*X(I)+Y(I)*Y(I))
23  WRITE(6,216)X(I),Y(I),TEST
216  FORMAT(1H ,3F15.8)
      IF(IND.LT.2)GO TO 22
C
C  CALCULATE OPEN LOOP AND CLOSED LOOP RESPONSE
C
      WRITE(6,208)
208  FORMAT(24H0 UNCONTROLLED RESPONSE//)
      TIME=0.
      K=0
      DO 11 I=1,N
11  X(I)=1.0
12  WRITE(6,209)TIME,K,(X(I),I=1,N)
209  FORMAT(1H0,F10.4,15,5X,1P6E15.5)
      CALL MATVEC(PHI,X,Y,N,N)
      DO 13 I=1,N
      X(I)=Y(I)
13  IF(X(I).GT.1.0E4)GO TO 14
      K=K+1
      TIME=TIME+DT
      IF(K.LT.41)GO TO 12
14  CONTINUE
      WRITE(6,210)
210  FORMAT(22H0 CONTROLLED RESPONSE//)
      TIME=0.
      K=0
      CALL MULTIP(DEL,FBM,R,N,M,N)
      DO 15 I=1,N
15  X(I)=1.0
16  WRITE(6,212)TIME,K,(X(I),I=1,N)
212  FORMAT(1H ,F10.4,15,3X,1P6E15.5)
      CALL MATVEC(R,X,U1,N,N)
      WRITE(6,211)(U1(I),I=1,M)
211  FORMAT(1H ,70X,1P3E15.5)
      CALL MATVEC(PHI,X,Y,N,N)
      DO 17 I=1,N
      X(I)=Y(I)-U1(I)
17  IF(X(I).GT.1.0E4)GO TO 18
      K=K+1
      TIME=TIME+DT
      IF(K.LT.41) GO TO 16
18  CONTINUE
22  CONTINUE
      RETURN
      END

```

```

SUBROUTINE RICATI(N,M,MAX,DT,EPS,FB,IND)
C
C KALMAN-ENGLAR 28-OCT-76
C
C THE CONTINUOUS DYNAMIC RICATI EQUATION IS SOLVED BY THE
C ALGORITHM OF KALMAN-ENGLAR(K & S, PAGE 249)
C
COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
1Q(6,6),R(6,6),XPLOT(6,200),UPLOT(6,200),DPLOT(6,200),
2UMAX(6),UMIN(6),XO(6)
DIMENSION V(6,6),D(6,6),E(6,6),F(6,6),G(6,6),FB(6,6),P(6,6),S(6,6)
1,WR(6),WI(6),Z(6,6),THETA(6,6),H(6,6)
DOUBLE PRECISION DET
WRITE(6,219)
219 FORMAT(40H1 SUBROUTINE RICATI(N,M,MAX,DT,EPS,IND)/)
WRITE(6,220)N,M,MAX,DT,EPS,IND
220 FORMAT(1H0,3I3,F10.4,E15.5,I3/)
IF(N.LT.3)GO TO 50
WRITE(6,311)
311 FORMAT(42H0 SYSTEM IS TOO LARGE FOR THE SUBROUTINES/)
RETURN
50 CONTINUE
DO1 I=1,N
DO1 J=1,N
H(I,J)=0.
P(I,J)=0.
IF(I.NE.J)GO TO 1
H(I,J)=1.0
P(I,J)=1.0
1 CONTINUE
IF(IND.LT.1)GO TO 5
WRITE(6,200)
200 FORMAT(20H0 SYSTEM MATRIX - A/)
DO32 I=1,N
32 WRITE(6,201)(A(I,J),J=1,N)
201 FORMAT(1H0,6F12.6)
WRITE(6,202)
202 FORMAT(21H0 DRIVING MATRIX - B/)
DO33 I=1,N
33 WRITE(6,201)(B(I,J),J=1,M)
WRITE(6,203)
203 FORMAT(25H0 MEASUREMENT MATRIX - H/)
DO34 I=1,N
34 WRITE(6,201)(H(I,J),J=1,N)
WRITE(6,204)
204 FORMAT(27H0 STATE PENALTY MATRIX - Q/)
DO35 I=1,N
35 WRITE(6,201)(Q(I,J),J=1,N)
WRITE(6,205)
205 FORMAT(29H0 CONTROL PENALTY MATRIX - R/)
DO36 I=1,M
36 WRITE(6,201)(R(I,J),J=1,M)
5 CONTINUE
IF(IND.LT.2)GO TO 2
C
C CALCULATE OPEN LOOP EIGENVALUES
C
DO 15 I=1,N
DO 15 J=1,N
15 V(I,J)=A(I,J)
CALL EIGVEC(6,N,V,WR,WI,S)
WRITE(6,218)
218 FORMAT(31H0 OPEN LOOP SYSTEM EIGENVALUES/
110X," REAL",10X," IMAGINARY"/)
DO 19 I=1,N
19 WRITE(6,217)WR(I),WI(I)
2 CONTINUE
C
C CONSTRUCT THE "Z" MATRIX
C
CALL TRANSP(B,V,N,M)
CALL INVERT(6,M,R,D,DET)
CALL MULTIP(D,V,FB,M,M,N)
CALL MULTIP(B,FB,V,N,M,N)
CALL TRANSP(H,D,N,N)
CALL MULTIP(D,Q,E,N,N,N)
CALL MULTIP(E,H,D,N,N,N)
CALL TRANSP(A,E,N,N)
DO 8 I=1,N
DO 8 J=1,N
Z(I,J)=-A(I,J)
Z(I,J+N)=V(I,J)
Z(I+N,J)=D(I,J)
8 Z(I+N,J+N)=E(I,J)

```

```

      IF(IND.LT.3)GO TO 3
      WRITE(6,207)
207  FORMAT(27H0 THE MAGNIFICENT Z MATRIX/)
      DO 7 I=1,2*N
      7  WRITE(6,206)(Z(I,J),J=1,2*N)
206  FORMAT(1H0,12F10.6)
      CONTINUE
3
C
C   CALCULATE THE TRANSITION MATRIX THETA
C   BY MATRIX EXPONENTIATION
C
      CALL EXPONT(2*N,Z,THETA,DT,EPS,IND)
C
C   PARTITION THE TRANSITION MATRIX INTO V,D,E & F
C
      DO 9 I=1,N
      DO 9 J=1,N
      V(I,J)=THETA(I,J)
      D(I,J)=THETA(I,J+N)
      E(I,J)=THETA(I+N,J)
      F(I,J)=THETA(I+N,J+N)
      9  CONTINUE
      IF(IND.LT.3)GO TO 4
      WRITE(6,300)DT
300  FORMAT(20H0 TIME INTERVAL IS ,F12.6/)
      WRITE(6,208)
208  FORMAT(29H0 INITIAL RICATTI MATRIX - P/)
      DO 10 I=1,N
      10  WRITE(6,201)(P(I,J),J=1,N)
4
C
C   START THE ITERATIVE SEQUENCE
C
      L=0
      14 SUM=0.
      CALL MULTIP(F,P,H,N,N,N)
      CALL ADDMAT(E,H,S,N,N)
      CALL MULTIP(D,P,H,N,N,N)
      CALL ADDMAT(V,H,G,N,N)
      CALL INVERT(G,N,G,H,DET)
      CALL MULTIP(S,H,G,N,N,N)
      L=L+1
      DO 11 I=1,N
      DO 12 J=1,N
      DIFF=ABS(P(I,J)-G(I,J))
      SUM=SUM+DIFF
      12 P(I,J)=G(I,J)
      IF(IND.LT.3)GO TO 11
      WRITE(6,209)(P(I,J),J=1,N)
209  FORMAT(30X,6F12.6)
      11 CONTINUE
      IF(IND.LT.2)GO TO 37
      WRITE(6,210) SUM
210  FORMAT(7H DIFF=,F16.10)
      37 CONTINUE
      IF(SUM.LT.EPS) GO TO 13
      IF(L.LE.MAX) GO TO 14
      WRITE(6,211) L
211  FORMAT(24H0 NO CONVERGENCE AFTER ,I3," ITERATIONS"/)
      GO TO 21
      13 WRITE(6,212) L
212  FORMAT(21H0 CONVERGENCE AFTER ,I3," ITERATIONS"/
      110X," STEADY-STATE RICATTI MATRIX - P"/)
      DO 16 I=1,N
      16 WRITE(6,213)(P(I,J),J=1,N)
213  FORMAT(1H0,10X,6F15.8)
C
C   CALCULATE STEADY-STATE FEEDBACK MATRIX
C
21  CALL MULTIP(FB,P,D,M,N,N)
      WRITE(6,214)
214  FORMAT(31H0 OPTIMAL FEEDBACK GAIN MATRIX/)
      DO 17 I=1,M
      DO 22 J=1,N
      17 FB(I,J)=D(I,J)
      22 WRITE(6,215)(FB(I,J),J=1,N)
215  FORMAT(1H0,10X,6F15.8)
      IF(IND.LT.2)GO TO 6
C
C   CALCULATE CLOSED LOOP MATRIX AND EIGENVALUES
C
      CALL MULTIP(B,D,E,N,M,N)
      CALL SUBMAT(A,E,G,N,N)
      CALL EIGVEC(G,N,G,WR,WI,S)
      WRITE(6,216)
216  FORMAT(33H0 CLOSED LOOP SYSTEM EIGENVALUES/
      110X," REAL",10X," IMAGINARY"/)
      DO 18 I=1,N
      18 WRITE(6,217)WR(I),WI(I)
217  FORMAT(1H0,5X,2F15.8)
6
      CONTINUE
      RETURN
      END

```

```

SUBROUTINE EXPONT(N,A,B,T,EPS,IND)
C      MATRIX EXPONENTIAL 28-OCT-76
C
C      DIMENSION A(6,6),B(6,6),G(6,6),E(6,6),F(6,6)
C      IF(IND.LT.2)GO TO 13
C      DO 12 I=1,N
12 WRITE(6,205)(A(I,J),J=1,N)
13 CONTINUE
C
C      CALCULATE MATRIX NORM
C
C      ANORM=0.
C      DO 1 J=1,N
C      AN=0.
C      DO 2 I=1,N
2 AN=AN+ABS(A(I,J))
1 IF(AN.GT.ANORM) ANORM=AN
C
C      TEST THAT T IS SMALL ENOUGH TO CONVERGE
C
3 IF(ANORM*T.LT.20.) GO TO 4
T=T/2.0
GO TO 3
4 WRITE(6,200)ANORM,T
200 FORMAT(16H0 NORM OF A IS ,F12.6," AND TIME INTERVAL IS",
1F10.4/)
C
C      SET UP INITIAL MATRICES B & G
C
C      DO 5 I=1,N
C      DO 5 J=1,N
C      B(I,J)=0.
C      G(I,J)=0.
C      IF(I.NE.J) GO TO 5
C      B(I,J)=1.0
C      G(I,J)=1.0
5 CONTINUE
C
C      CALCULATE TERMS IN MATRIX SERIES WITH COUNTER
C
P=0.0
10 P=P+1.0
DO 6 I=1,N
DO 6 J=1,N
6 E(I,J)=A(I,J)*T/P
CALL MULTIP(G,E,F,N,N,N)
GABS=0.
DO 7 I=1,N
DO 7 J=1,N
G(I,J)=F(I,J)
IF(ABS(G(I,J)).GT.GABS) GABS=ABS(G(I,J))
7 B(I,J)=B(I,J)+G(I,J)
C
C      TEST FOR CONVERGENCE
C
M=P
IF(IND.LT.2)GO TO 14
WRITE(6,201)M,GABS
201 FORMAT(32H0 MAXIMUM TERM IN SERIES AFTER ,I3,
1" ITERATIONS IS ",F12.8/" MATRIX EXPONENTIAL IS "/)
DO 8 I=1,N
8 WRITE(6,202)(B(I,J),J=1,N)
202 FORMAT(1H ,10X,8F12.6)
14 CONTINUE
IF(GABS.LT.EPS) GO TO 9
IF(M.LE.100) GO TO 10
WRITE(6,203)
203 FORMAT(23H0 CONVERGENCE TOO SLOW)
RETURN
9 WRITE(6,204) M
204 FORMAT(21H0 CONVERGENCE AFTER ,I4," ITERATIONS"/
110X," TRANSITION MATRIX"/)
DO 11 I=1,N
11 WRITE(6,205)(B(I,J),J=1,N)
205 FORMAT(5X,8F12.6)
RETURN
END

```

```

SUBROUTINE NEWEL2(N,M,K,PHI,DEL,PSI,Q,R,KFB,KFF,DT,IND)
C
C STEADY STATE FEEDBACK AND FEEDFORWARD GAIN MATRICES SOLVED
C BY DYNAMIC PROGRAMMING (NEWELL,1971). ROUTINE READ PENALTY
C MATRICES - STATE S+Q, AND CONTROL -R. CLOSED LOOP EIGEN VALUES
C ARE CALCULATED. IND=0- ONLY FINAL F/B + F/F MATRICES ARE
C WRITTEN; IND=1- F/B+ F/F MATRICES WRITTEN AT EACH
C ITERATION; IND=2- F/B, F/F + P MATRICES PRINTED
C AT EACH ITERATION. DEL(N,M) IS THE DRIVING MATRIX
C AND PSI(N,K) THE DISTURBANCE MATRIX.
C
C DIMENSION PHI(6,6),DEL(6,6),PSI(6,6),S(6,6),Q(6,6),R(6,6),P(6,6),T
1(6,6),A(6,6),B(6,6),C(6,6),D(6,6),E(6,6),F(6,6),G(6,6),H(6,6)
2,WR(6),VI(6)
REAL KFB(6,6),KFF(6,6)
DOUBLE PRECISION DETERM
C
C INITIALISE S MATRIX
DO 32 IJ=1,N
DO 32 JI=1,N
32 S(IJ,JI)=Q(IJ,JI)
IF(IND.LE.1)GO TO 31
C
C WRITE OUT ALL THE SYSTEM MATRICES
C
WRITE(6,200)DT
200 FORMAT(37H1 EVALUATION OF FEEDBACK GAIN MATRIX//
15X,"TRANSITION MATRIX - PHI(",F6.4,")"/)
DO 1 I=1,N
1 WRITE(6,201)(PHI(I,J),J=1,N)
WRITE(6,202)DT
202 FORMAT(24H0 DRIVING MATRIX - DEL(",F6.4,")"/)
DO 2 I=1,N
2 WRITE(6,201)(DEL(I,J),J=1,M)
WRITE(6,203)
203 FORMAT(29H0 PENALTY MARICES - S,Q & R/)
DO 3 I=1,N
3 WRITE(6,201)(S(I,J),J=1,N)
DO 4 I=1,N
4 WRITE(6,201)(Q(I,J),J=1,N)
DO 5 I=1,M
5 WRITE(6,201)(R(I,J),J=1,M)
201 FORMAT(1H0,5X,6F12.6)
31 CONTINUE
KK=0
CALL ADDMAT(Q,S,P,N,N)
DO 7 I=1,N
DO 7 J=1,K
7 T(I,J)=0.
CALL TRANSP(DEL,A,N,M)
IF(IND.LT.2)GO TO 8
DO 9 I=1,N
9 WRITE(6,204)(P(I,J),J=1,N)
204 FORMAT(1H ,70X,"P",6F9.4)
DO 10 I=1,N
10 WRITE(6,205)(T(I,J),J=1,K)
205 FORMAT(1H ,70X,"T",6F9.4)
8 KK=1
C
C START ITERATIVE ROUTINE
C
11 CONTINUE
IKON=0
CALL MULTIP(A,P,B,M,N,N)
CALL MULTIP(B,DEL,D,M,N,M)
CALL ADDMAT(D,R,E,M,M)
IF(M.GT.1)GO TO 12
D(1,1)=1.0/E(1,1)
GO TO 13
12 CALL INVERT(6,M,E,D,DETERM)
13 CALL MULTIP(B,PHI,E,M,N,N)
CALL MULTIP(D,E,F,M,M,N)
SUM=0.
DO 14 I=1,M
DO 15 J=1,N
DIFF=ABS(KFB(I,J)+F(I,J))
KFB(I,J)=-F(I,J)
15 SUM=SUM+DIFF
IF(IND.LT.1)GO TO 14
WRITE(6,206)KK,(KFB(I,J),J=1,N)
206 FORMAT(1H ,13," KFB",3X,6F12.6)

```

```

14 CONTINUE
   IF(SUM.LT.1.0E-9)IKON=1
   CALL MULTIP(B,PSI,E,M,N,K)
   CALL MULTIP(A,T,F,M,N,K)
   CALL ADDMAT(E,F,B,M,K)
   CALL MULTIP(D,B,E,M,M,K)
   SUM=0.
   DO 16 I=1,M
   DO 17 J=1,K
   DIFF=ABS(KFF(I,J)+E(I,J))
   KFF(I,J)=-E(I,J)
17 SUM=SUM+DIFF
   IF(IND.LT.1)GO TO 16
   WRITE(6,207)KK,(KFF(I,J),J=1,K)
207 FORMAT(1H,13," KFF",3X,6F12.6)
16 CONTINUE
   IF(SUM.LT.1.0E-9)IKON=IKON+1
C
C   CALCULATE NEW CLOSED LOOP MATRIX -C
C
   CALL MULTIP(DEL,KFB,B,N,M,N)
   CALL ADDMAT(PHI,B,C,N,N)
   CALL TRANSP(C,D,N,N)
   IF(IKON.GT.1)GO TO 20
C
C   CALCULATE NEW P+T MATRICES
C
   CALL MULTIP(D,P,E,N,N,N)
   CALL MULTIP(E,C,F,N,N,N)
   CALL TRANSP(KFB,E,H,N)
   CALL MULTIP(E,R,G,N,M,M)
   CALL ADDMAT(F,H,E,N,N)
   CALL ADDMAT(E,Q,F,N,N)
   CALL MULTIP(D,T,B,N,N,K)
   CALL MULTIP(C,P,D,N,N,N)
   CALL MULTIP(DEL,KFF,E,N,M,K)
   CALL ADDMAT(E,PSI,H,N,K)
   CALL MULTIP(D,H,E,N,N,K)
   CALL MULTIP(G,KFF,D,N,M,K)
   CALL ADDMAT(B,E,H,N,K)
   CALL ADDMAT(H,D,T,N,K)
   DO 18 I=1,H
   DO 19 J=1,N
19 P(I,J)=F(I,J)
   IF(IND.LT.2)GO TO 18
   WRITE(6,204)(P(I,J),J=1,N)
18 CONTINUE
   IF(IND.LT.2)GO TO 21
   DO 22 I=1,N
22 WRITE(6,205)(T(I,J),J=1,K)
21 KK=KK+1
   IF(KK.LE.100)GO TO 11
C
C   CONVERGENCE NOT ACHIEVED
C
   WRITE(6,208)KK
208 FORMAT(24H0 NO CONVERGENCE AFTER ,13," ITERATIONS"/
15X," CURRENT FEEDBACK MATRIX -")
   DO 23 I=1,H
23 WRITE(6,209)(KFB(I,J),J=1,N)
209 FORMAT(1H0,10X,6F12.8)
   GO TO 24
C
C   CONVERGENCE ACHIEVED]
C
20 WRITE(6,210)KK
210 FORMAT(21H0 CONVERGENCE AFTER ,13," ITERATIONS"/
15X," ULTIMATE FEEDBACK MATRIX -")
   DO 25 I=1,M
25 WRITE(6,209)(KFB(I,J),J=1,N)
   WRITE(6,211)
211 FORMAT(34H0 ULTIMATE FEED FORWARD MATRIX -)
   DO 27 I=1,M
27 WRITE(6,209)(KFF(I,J),J=1,K)
C
C   CALCULATE EIGEN VALUES OF CLOSED LOOP MATRIX
C
24 CALL EIGVEC(6,N,C,WR,WI,A)
   WRITE(6,212)
212 FORMAT(36H0 EIGENVALUES OF CLOSED LOOP SYSTEM/
110X,"REAL",10X,"IMAGINARY",10X,"STABILITY CRITERION")
   DO 26 I=1,N
   CRIT=SQRT(WR(I)*WR(I)+WI(I)*WI(I))
26 WRITE(6,213)WR(I),WI(I),CRIT
213 FORMAT(1H0,5X,F12.6,10X,F12.6,10X,F12.6)
   RETURN
   END

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C      EXOREACTOR-2-2-2 20-FEB-77
C
C      A SECOND ORDER NON-LINEAR REACTOR IS SIMULATED AND TESTED
C      WITH A LINEAR FEEDBACK CONTROL ALGORITHM.
C
C      IND1 CONTROLS DIAGNOSTIC PRINTOUT.
C      IND2 CONTROLS THE CALCULATION OF EIGENVALUES.
C      IND3=1, CONTINUOUS SIMULATION; IND3=2, BOTH;
C      IND3=4, DISCRETE SIMULATION ONLY.
C      IND4 STORES TRAJECTORY FOR PLOTTING : 3, CONTINUOUS,
C                                          4, DISCRETE.
C
C      COMMON/COMP/A(6,6),B(6,6),C(6,6),PHI(6,6),DEL(6,6),PSI(6,6),
1Q(6,6),R(6,6),X(6,6),X(6,6),X(6,6),X(6,6),X(6,6),X(6,6),X(6,6),
2UMAX(6),UMIN(6),XO(6)
C      DIMENSION X(6),X1(6),V(6),U(6),D(6)
C      DIMENSION WR(6),WI(6)
C      REAL KFBC(6,6),KFBD(6,6),LIMIT,NOISE,INTEGR
C      COMMON DT,NN,I
C
C      READ PROCESS PARAMETERS
C
C      1 READ(5,100)N,M,K,IND1,IND2,IND3,IND4,DT,PRI,FINT,NPT
C
100 FORMAT (7I2,3F6.4,I3)
C      READ(5,101)CIS,CS,TIS,TS,FS
C      READ(5,101)UA,DELH,ROW,CP,VOL
C      READ(5,101)E,RG
101 FORMAT(6F10.5)
C
C      CALCULATE REACTION RATE CONSTANT AND STEADY STATE
C      COOLANT TEMPERATURE
C
C      TAU=VOL/FS
C      AND=FS/VOL
C      ERTS=E/(RG*TS)
C      EERTS=EXP(-ERTS)
C      RATE=AND*(CIS-CS)/(CS*CS*EERTS)
C      F1=ROW*CP*(TIS-TS)
C      F2=DELH*(CIS-CS)
C      TWS=TS-FS*(F1+F2)/UA
C      ERG=E/RG
C
C      CONSTRUCT LINEAR MODEL BASED ON STEADY-STATE VALUES.
C
C      A(1,1)=-AND*(2.0*CIS-CS)/CS
C      A(1,2)=-AND*ERTS*(CIS-CS)/CS
C      A(2,1)=AND*2.0*DELH*(CIS-CS)/(ROW*CP*TS)
C      A(2,2)=-AND-UA/(ROW*VOL*CP)+AND*DELH*ERTS*(CIS-CS)/(ROW*CP*TS)
C      B(1,1)=AND*(CIS-CS)/CS
C      B(1,2)=0.0
C      B(2,1)=AND*(TIS-TS)/TS
C      B(2,2)=UA*TWS/(ROW*VOL*CP*TS)
C      C(1,1)=AND*CIS/CS
C      C(1,2)=0.0
C      C(2,1)=0.0
C      C(2,2)=AND*TIS/TS
C      WRITE(6,200)N,M,K,DT,FINT
C
200 FORMAT(42H1 OPTIMAL CONTROL OF SECOND-ORDER REACTOR//
1" NUMBER OF MEASURED STATES IS ",I2//
2" NUMBER OF CONTROL VARIABLES IS ",I2//
3" NUMBER OF DISTURBANCE VARIABLES IS ",I2//
4" INTEGRATION STEPSIZE IS ",F6.3," WITH PROCESS DURATION ",F6.2//)
C      WRITE(6,201)CIS,CS,TIS,TS,TWS,FS,TAU,RATE,UA,DELH
201 FORMAT(34H0 STEADY-STATE REACTOR PARAMETERS//
1" INLET CONC. PRODUCT CONC. INLET TEMP. PRODUCT TEMP.
2" COOLANT TEMP."//2X,1P5E15.5// " FLOW RATE SPACE TIME
3" RATE CONSTANT HT TRANSF TERM DELH"//1X,1P5E15.5//)
C      WRITE(6,220)ERG,ERTS,EERTS
220 FORMAT(33H0 RATIO ERTS EXP(-ERTS)//1X,1P3E15.5//)
C      WRITE(6,218)
218 FORMAT(19H0 LINEARISED MODEL//)
C      DO 36 J=1,2
36 WRITE(6,202)A(J,1),A(J,2),B(J,1),B(J,2),C(J,1),C(J,2)
202 FORMAT(1H0,2F10.5,5X,2F10.5,5X,2F10.5)
C      IF(IND2.LT.2)GO TO 2
C      DO 38 IJ=1,N
C      DO 38 JI=1,N
38 PSI(IJ,JI)=A(IJ,JI)
C      CALL EIGVEC(6,N,PSI,WR,WI,DEL)
C      WRITE(6,216)
216 FORMAT(35H0 CONTINUOUS OPEN-LOOP EIGENVALUES//)
C      DO 37 IJ=1,N
37 WRITE(6,217)WR(IJ),WI(IJ)
217 FORMAT(1H0,2F15.5)

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WRITE(6,401)
401 FORMAT(39H0 COMPARISON OF LINEAR AND REAL MODELS/)
X(1)=0.01
X(2)=0.01
CALL MATVEC(A,X,V,2,2)
WRITE(6,400)V(1),V(2)
400 FORMAT(1H0,2F20.10/)
D(1)=0.0
D(2)=0.0
U(1)=0.0
U(2)=0.0
X(1)=0.01
X(2)=0.01
F1=CIS*(1.0+D(1))/CS-(1.0+X(1))
F2=E*X(2)/(RG*TS*(1.0+X(2)))
F3=EXP(F2)
F2=(CIS-CS)*(1.0+X(1))*(1.0+X(1))/CS
ARG1=AND*((1.0+U(1))*F1-F2*F3)
F1=TIS*(1.0+D(2))/TS-(1.0+X(2))
F2=(1.0+X(2))-TWS*(1.0+U(2))/TS
F4=DELH*(CIS-CS)*(1.0+X(1))*(1.0+X(1))/(ROW*CP*TS)
ARG2=AND*((1.0+U(1))*F1+F3*F4)-UA*F2/(ROW*VOL*CP)
WRITE(6,400)ARG1,ARG2
V(1)=0.0
V(2)=0.0

C
C ENTER PROCESS CONTROL DATA
C
2 READ(5,102)T
102 FORMAT (F6.4)
READ (5,/)((Q(I,J),J=1,2),I=1,2)
3 READ (5,/)R(1,1),R(1,2),R(2,1),R(2,2)
DO 4 I=1,2
4 READ(5,/)UMIN(I),UMAX(I)
5 READ(5,/)XO(1),XO(2)
WRITE (6,203)T,Q(1,1),Q(1,2),Q(2,1),Q(2,2),R(1,1),R(1,2),
1R(2,1),R(2,2),UMIN(1),UMAX(1),UMIN(2),UMAX(2)
203 FORMAT(37H0 CONTROL PARAMETERS AND CONSTRAINTS/
1" DISCRETE TIME INTERVAL IS ",F6.4/
2" STATE PENALTY MATRIX = Q"/ 10X,2F20.4/ 10X,2F20.4/
3" CONTROL PENALTY MATRIX = R"/ 10X,2F20.4/10X,2F20.4/
4" CONTROL CONSTRAINTS"/(10X,"UMIN = "F10.4," UMAX = ",F10.4/))

C
C CALCULATE DISCRETE LINEAR MODEL
C GO TO 8

C
IF(IND3.LT.2) GO TO 6
CALL PHDEL(N,M,K,T,IND1)

C
C CALCULATE THE DISCRETE FEEDBACK MATRIX
C
CALL CONTR1(N,M,PHI,DEL,Q,R,KFBD,T,IND1)
6 CONTINUE
IF(IND3.GT.2) GO TO 7

C
C CALCULATE FEEDBACK MATRIX FOR CONTINUOUS SYSTEM.
C
CALL RICAT1(N,M,200,1.0,1.0E-6,KFBC,IND1)
7 CONTINUE
GO TO 5

C
C
C RUN THE NON-LINEAR PROCESS WITH FEEDBACK CONTROL.
C IND3 CHOOSES CONTINUOUS,DISCRETE OR BOTH.
C IND4 SET TO 3 OR 4 STORES THE CONTINUOUS OR
C DISCRETE TRAJECTORIES RESPECTIVELY FOR PLOTTING.
C
IF(IND3.GT.2) GO TO 8
TNPT=NPT
PLTI=FINT/TNPT
WRITE(6,204)KFBC(1,1),KFBC(1,2),KFBC(2,1),KFBC(2,2),XO(1),XO(2),
1UMIN(1),UMAX(1),UMIN(2),UMAX(2)
204 FORMAT(43H1 PROCESS RUNNING UNDER CONTINUOUS CONTROL//
1" WITH FEEDBACK MATRIX KFBC = ",2F20.6/30X,2F20.6//
2" INITIAL CONDITION XO=(",2F8.4,")"/
3" AND CONTROL CONSTRAINTS",(" UMIN=",F8.4," UMAX=",F8.4/))
X(1)=XO(1)
X(2)=XO(2)
COST=0.0
TIME=0.0
PRT=0.0
PLTT=0.0
PLTNUM=1.0

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NN=0
WRITE(6,205)
205 FORMAT(42HO TIME COST STATE VARIABLES//)
WRITE(6,206)TIME,COST,X(1),X(2),D(1),D(2),U(1),U(2)
206 FORMAT(F8.2,1P7E15.5)
9 CONTINUE
X1(1)=X(1)
X1(2)=X(2)
C1 = COST
DO 10 I=1,2
DUM=NOISE(0.0,1.0,0.0,200)
DUM1=RAMP(0.25,TIME,1.0,0.025,20.0)
DUM2=RAMP(0.25,TIME,1.0,0.025,20.0)
IF(DUM.GE.0.0)GO TO 85
D(1)=DUM1
GO TO 86
85 D(1)=-DUM2
86 CONTINUE
D(2)=SIN(3.0*TIME)*DUM2*2.5
CALL MATVEC(KFBC,X,V,2,2)
DO 34 J=1,2
34 U(J)=-LIMIT(V(J),UMAX(J),UMIN(J))
IF(IND1.LT.2) GO TO 11
IF(TIME.GT.1.0)GO TO 11
WRITE(6,207)U(1),U(2)
207 FORMAT(83X,1P2E15.5)
11 CONTINUE
IF(1.LT.2) GO TO 12
IF(IND4.NE.3) GO TO 12
IF(TIME.LT.(PLTT-0.0001)) GO TO 12
XPLOT(1,PLTNUM)=X(1)
XPLOT(2,PLTNUM)=X(2)
UPLOT(1,PLTNUM)=U(1)
UPLOT(2,PLTNUM)=U(2)
PLTT=PLTT+PLTI
12 CONTINUE
F1=CIS*(1.0+D(1))/CS=(1.0+X(1))
F2=E*X(2)/(RG*TS*(1.0+X(2)))
F3=EXP(F2)
F2=(CIS-CS)*(1.0+X(1))*(1.0+X(1))/CS
ARG1=AND*((1.0+U(1))*F1-F2*F3)
F1=TIS*(1.0+D(2))/TS=(1.0+X(2))
F2=(1.0+X(2))-TWS*(1.0+U(2))/TS
F4=DELH*(CIS-CS)*(1.0+X(1))*(1.0+X(1))/(ROW*CP*TS)
ARG2=AND*((1.0+U(1))*F1+F3*F4)-UA*F2/(ROW*VOL*CP)
CALL QDRTC(X,Q,SQ,2,1.0)
CALL QDRTC(U,R,SR,2,1.0)
ARG3=SQ+SR
X(1)=INTEGR(1,XO(1),ARG1,X1(1))
X(2)=INTEGR(2,XO(2),ARG2,X1(2))
COST=INTEGR(3,0.0,ARG3,C1)
10 CONTINUE
PRICE=PRICE+ARG3*DT
IF(ABS(X(2)).GT.10.0) GO TO 15
IF(NN.LE.0) GO TO 13
TIME=TIME+DT
IF(TIME.LT.(PRT-0.0001)) GO TO 14
13 WRITE(6,206)TIME,COST,X(1),X(2),D(1),D(2),U(1),U(2)
PRT=PRT+PRI
14 NN=NN+1
IF(TIME.LT.(FINT-0.001)) GO TO 9
XPLOT(1,PLTNUM)=X(1)
XPLOT(2,PLTNUM)=X(2)
DPLOT(1,PLTNUM)=D(1)
DPLOT(2,PLTNUM)=D(2)
UPLOT(1,PLTNUM)=U(1)
UPLOT(2,PLTNUM)=U(2)
SOSQ=X(1)*X(1)+X(2)*X(2)
XEND=SQRT(SOSQ)
WRITE(6,208)TIME,COST,PRICE,XEND
208 FORMAT(31HO PROCESS COMPLETED AT TIME = ,F8.2//
1" COST=",1PE15.5,"PRICE=",1PE15.5,"XEND=",1PE15.5)
GO TO 16
15 WRITE(6,209)TIME,COST,PRICE,XEND
209 FORMAT(38HO EMERGENCY PLANT SHUTDOWN AT TIME = ,F8.2//
1" COST=",1PE15.5,"PRICE=",1PE15.5,"XEND=",1PE15.5/)
GO TO 19
16 CONTINUE
IF(IND4.NE.3) GO TO 19
CALL PLOTTR(N,M,T,FINT,2.5,3.5,8.0,IND4)
19 CONTINUE
8 CONTINUE

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C
C
C      RUN DISCRETE PROCESS.
      IF(IND3.LT.2) GO TO 20
      WRITE(6,210)KFBD(1,1),KFBD(1,2),KFBD(2,1),KFBD(2,2),X0(1),X0(2),
1      UMIN(1),UMAX(1),UMIN(2),UMAX(2)
210  FORMAT(33H1 REACTOR UNDER DISCRETE CONTROL//
1      " WITH FEEDBACK MATRIX KFBD = ",2F15.6/31X,2F15.6//
2      " INITIAL CONDITIONS X0=(",2F8.4,")"//
3      " AND CONTROL CONSTRAINTS "(" UMIN=",F8.4," UMAX=",F8.4/))
      TNPT=NPT
      PLTI=FINT/TNPT
C
C
C      CORRECT THE PRINTOUT INTERVAL TO FIT CONTROL INTERVAL
      IF(PRI-T)21,22,23
21  PN=T/PRI
      NP=PN
      PRTI=T/NP
      GO TO 24
22  PRTI=T
      NP=1
      GO TO 24
23  PN=PRI/T
      NP=PN
      PRTI=T*NP
24  CONTINUE
      WRITE(6,215)
215  FORMAT(1H0," TIME KOUNT COST STATE VARIABLES AND CONTROLS")
      D(1)=0.0
      D(2)=0.0
      U(1)=0.0
      U(2)=0.0
      X(1)=X0(1)
      X(2)=X0(2)
      CNT=0.0
      TIME=0.0
      COST=0.0
      NPLT=1
      PLTT=0.0
      PRT=0.0
      PRICE=0.0
      NN=0
      KOUNT=0
      WRITE(6,211)TIME,KOUNT,COST,X(1),X(2),D(1),D(2)
211  FORMAT(F8.2,15,1P5E15.5)
25  CONTINUE
      X1(1)=X(1)
      X1(2)=X(2)
      C1=COST
C
C
C      STORE VARIABLES FOR PLOTTING
      IF(IND4.NE.4) GO TO 26
      IF(TIME.LT.(PLTT-0.0001)) GO TO 26
      PLTT=PLTT+PLTI
      XPLOT(1,NPLT)=X(1)
      XPLOT(2,NPLT)=X(2)
      DPLOT(1,PLTNUM)=D(1)
      DPLOT(2,PLTNUM)=D(2)
      NPLT=NPLT+1
26  CONTINUE
      IF(TIME.LT.(CNT-0.0001)) GO TO 27
      CNT=CNT+T
      KOUNT=KOUNT+1
      CALL MATVEC(KFBD,X,V,2,2)
      DO 35 J=1,2
      U(J)=-LIMIT(V(J),UMAX(J),UMIN(J))
35  CONTINUE
      IF(TIME.GT.(PRT+PRI)) GO TO 28
      IF(TIME.GT.(PRT+T)) GO TO 28
      WRITE(6,212)U(1),U(2)
212  FORMAT(89X,1P2E15.5)

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C      STORE CONTROL VARIABLE FOR PLOTTING
C
28 IF(IND4.NE.4) GO TO 29
  UPLOT(1,KOUNT)=U(1)
  UPLOT(2,KOUNT)=U(2)
29 CONTINUE
  CALL QUDRTC(X,Q,SQ,2,T)
  CALL QUDRTC(U,R,SR,2,T)
  PRICE=PRICE+SQ+SR
  IF(IND1.LT.2) GO TO 27
  IF(KOUNT.GT.(2*NP)) GO TO 27
  WRITE(6,213)TIME,KOUNT,COST,X(1),X(2),D(1),D(2),U(1),U(2)
213 FORMAT(F8.2,I5,1P7E15.5)
27 CONTINUE
  DO 30 I=1,2
    DUM=NOISE(0.0,1.0,0.0,200)
    DUM1=RAMP(0.25,TIME,1.0,0.025,20.0)
    DUM2=RAMP(0.25,TIME,1.0,0.025,20.0)
    IF(DUM.LE.0.0)GO TO 95
    D(1)=DUM1
    GO TO 96
95 D(1)=-0.65*DUM2
96 CONTINUE
    D(2)=SIN(3.0*TIME)*DUM2*2.5
    F1=CIS*(1.0+D(1))/CS-(1.0+X(1))
    F2=E*X(2)/(RG*TS*(1.0+X(2)))
    F3=EXP(F2)
    F2=(CIS-CS)*(1.0+X(1))*(1.0+X(1))/CS
    ARG1=AND*((1.0+U(1))*F1-F2*F3)
    F1=TIS*(1.0+D(2))/TS-(1.0+X(2))
    F2=(1.0+X(2))-TWS*(1.0+U(2))/TS
    F4=DELH*(CIS-CS)*(1.0+X(1))*(1.0+X(1))/(ROW*CP*TS)
    ARG2=AND*((1.0+U(1))*F1+F3*F4)-UA*F2/(ROW*VOL*CP)
    CALL QUDRTC(X,Q,SQ,2,1.0)
    CALL QUDRTC(U,R,SR,2,1.0)
    ARG3=SQ+SR
    X(1)=INTEGR(1,X0(1),ARG1,X1(1))
    X(2)=INTEGR(2,X0(2),ARG2,X1(2))
    COST=INTEGR(3,0.0,ARG3,C1)
    IF(ABS(X(2)).GT.10.0)GO TO 40
30 CONTINUE
    IF(NN.LE.0) GO TO 31
    TIME=TIME+DT
    IF(TIME.LT.(PRT-0.0001)) GO TO 32
31 WRITE(6,211)TIME,KOUNT,COST,X(1),X(2),D(1),D(2)
    PRT=PRT+PRTI
32 NN=NN+1
    IF(TIME.LT.(FINT-0.001)) GO TO 25
    XPLOT(1,NPLT)=X(1)
    XPLOT(2,NPLT)=X(2)
    DPLOT(1,NPLT)=D(1)
    DPLOT(2,NPLT)=D(2)
    SOSQ=X(1)*X(1)+X(2)*X(2)
    XEND=SQRT(SOSQ)
    WRITE(6,214)TIME,COST,PRICE,XEND
214 FORMAT(31H0 PROCESS COMPLETED AT TIME = ,F10.2//
  1" COST=",F20.6," PRICE=",F20.6", XEND="1PE15.5)
    GO TO 39
40 WRITE(6,209)TIME,COST,PRICE,XEND
    GO TO 33
39 IF(IND4.NE.4) GO TO 33
  CALL PLOTTR(N,M,T,FINT,2.5,3.0,8.0,IND4)
33 CONTINUE
20 CONTINUE
  GO TO 5
END

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